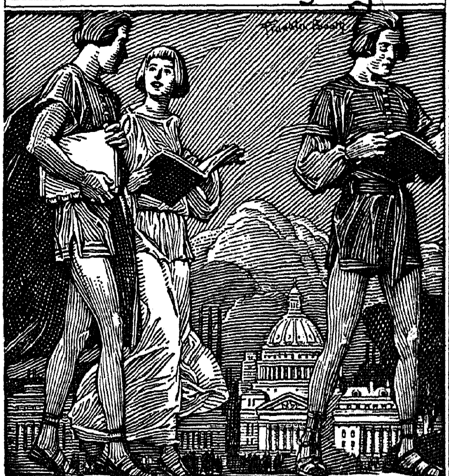

INTRODUCTION

TO

ELECTRIC TRANSIENTS

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ELECTRIC TRANSIENTS

BY

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PREFACE

This introductory text is prepared primarily for class use. A sincere attempt is made to reach the student and to provide him with the background which is necessary for advanced study in the field. Particular attention has been paid to the pedagogical arrangement of the subject matter. Both the d–c section and the a–c section begin with simple circuit configurations and proceed to the more complex. Natural circuit behavior is stressed throughout the text because transient disturbances in general are merely the manifestations of the "natural behavior" of the particular circuit. Experience has shown that this procedure greatly simplifies the study of transient phenomena in alternating-current circuits.

The method of presentation employed is to consider each type of transient under three distinct headings, namely,

1. Physical Considerations.
2. Mathematical Analysis.
3. Oscillographic Verification.

Under "Physical Considerations," an attempt is made to present the physical aspects of the circuit conditions that are under discussion. The student is encouraged to analyze the problem in terms of the principles and laws with which he is already familiar. This is done to strengthen his analytic ability and his understanding of the basic laws governing the behavior of electrical circuits. It also insures that each student has a reasonably clear conception of the electrical situation before the mathematical treatment is undertaken.

Under "Mathematical Analysis," the problem is treated further as a mathematical development. Given the fundamental equilibrium equations, the results are derived in accordance with the laws of conventional mathematics, usually involving the use and solution of some form of differential equation. In addition to the conventional solution there is also given a solution in terms of Heaviside's operational calculus. An introduction to this form of mathematics is included in the Appendix, but the actual operational solution of each circuit problem is given alongside the conventional solution. This provides a convenient means of becoming acquainted with Heaviside's operational calculus and at the same time affords opportunity to observe its advantages or disad-

vantages over the conventional calculus. Furthermore, it prepares the student for the mastery of the concepts of transients in time and space in which the Heaviside method has certain advantages. However, the entire mathematical treatment is such that the major portion of the text may be pursued even though the operational solutions are omitted. Only in those cases where the conventional method of attack becomes too laborious is the operational solution the only one given.

Actual oscillograms accompany all the major discussions and analyses. It is believed that experimental verification of the laws governing transient electrical phenomena cannot be over-emphasized in an introductory course. The validity and preciseness of the current, voltage, and power equations are best appreciated by taking oscillograms in circuits in which the circuit parameters are known and comparing these oscillographic records with the plotted solutions of the derived expressions. The remarkable agreement that will be found to exist between the calculated curves and the photographic records for a given set of circuit parameters helps to remove any skepticism which the student may hold toward the derived expressions.

The subject of "power" transients has been given a prominent place throughout the text. This more or less neglected aspect of electric transient phenomena is considered to be important since electrical engineering is fundamentally a study of electrical energy, and its rate of transfer which is power. The rate at which energy is generated, absorbed, or stored in electrical circuits and machines is often a problem of major interest. It is hoped that the treatment of power transients together with the numerous power oscillograms will constitute a certain contribution to the subject.

Commercial examples of electric transients are, of course, cited from time to time throughout the text. To give a broader view of the field, however, Chapter X has been given over largely to the presentation of some of the common but nevertheless important electrical machinery transients.

The authors wish to express their thanks to the students who have helped with the preparation of the oscillograms and numerical examples. Acknowledgment and thanks are also due to Dr. Allen Craig of the Mathematics Department at The State University of Iowa for reviewing the mathematical appendix, and to Professor H. R. Reed of the Electrical Engineering Department at the Michigan College of Mining and Technology for his constructive criticisms of the text material.

E. B. Kurtz
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Iowa City, Iowa
January 7, 1935

TO THE STUDENT

NATURE OF THE SUBJECT MATTER

The subject of "Transients" embraces a very wide variety of physical phenomena. A course in *Electric Transients* concerns itself chiefly with the study of electric circuits and machines during their *transient* periods. In the usual undergraduate courses, formulas and expressions are developed which are valid only during the periods in which the circuits and machines are in their so-called "steady states." These expressions carry with them the tacit assumption that the electrical circuits involved are in a steady-state condition of operation, for example, that the currents have become constant in the case of direct-current circuits or have become periodic functions of time in the case of alternating-current circuits.

The steady-state solution of a circuit problem is only one part of the complete solution, for immediately after the establishment of the circuit the currents and the potentials have not, in general, settled into steady-state conditions. A certain lapse of time is required for the electrical conditions to adjust themselves to their ultimate steady-state modes of variation. This time interval is called the transient period. In general, any switching operation or other sudden circuit disturbance will be the occasion of a non-steady-state or *transient* period. The investigations of currents, component voltages, and associated phenomena during transient periods comprise the study of *Electric Transients* or of *Transient Electric Phenomena*, as the study is sometimes called.

BASIC PHYSICAL LAWS AND CONCEPTS

Transient circuit theory as well as steady-state circuit theory finds its basic support in the *law of the conservation of energy*, the common meeting-place of the various sciences. Under dynamic conditions of operation, a given portion of an electric circuit receives energy from some driving or supplying source. In general, a branch of an electric circuit is capable of transforming the energy that it receives into three distinct forms, namely,

1. Electromagnetic energy (kinetic in form).
2. Electrostatic energy (potential in form).
3. Heat energy.

The sum of the three components over a particular interval of time (*energy transformed*) is, by the law of the conservation of energy, equal to the energy *delivered* to the branch during this interval. Neglecting the case of energy propagated into space as a result of high-frequency phenomena and assuming that no energy is stored in the branch at $t = 0$, the energy distribution may be expressed as follows:

Energy delivered to the branch		Energy transformed into heat		Electrical kinetic energy at time, t_1 (i_1 is the current at time t_1)		Electrical potential energy at time, t_1 (e_{c1} is the condenser voltage at time t_1)
$\int_0^{t_1} Ei\,dt$	$=$	$\int_0^{t_1} Ri^2\,dt$	$+$	$\int_0^{i_1} Li\,di$	$+$	$\int_0^{e_{c1}} Ce_c\,de_c$

The above energy equation applies to a branch of an electric circuit which possesses resistance (R), self-inductance (L), and series capacitance (C).

By taking the derivative of each side of the above energy equation with respect to time, the power relations in the branch are at once obtainable. The power equation is:

Power delivered to the branch		Power loss in resistance		Rate of energy storage in the magnetic field		Rate of energy storage in the electric field
Ei	$=$	Ri^2	$+$	$Li\dfrac{di}{dt}$	$+$	$e_c i$

The fundamental equation for voltage equilibrium in the branch follows directly from the above power equation. Dividing through by i:

Voltage applied to the branch		Resistive counter-voltage		Inductive counter-voltage		Capacitive counter-voltage
E	$=$	Ri	$+$	$L\dfrac{di}{dt}$	$+$	e_c

The above equation is one form of Kirchhoff's emf law which taken together with Kirchhoff's current law and Lenz's law form the basis of modern electrical circuit analysis. These laws are in no sense ultimate explanations of the phenomena involved. They merely state physical facts which have been experimentally determined and which have proved their usefulness in quantitative analysis. They are taken as the starting-points in the mathematical analyses of the various circuits treated in this text. The student will profit by inquiring beyond these so-called basic laws, seeking as well as he may explanations of the electrical phenomena in terms of the "electron and proton" theory of matter.

TYPE OF MATHEMATICS INVOLVED

The equations given in the above section are typically characteristic of those encountered throughout the text. They are differential equations of a very simple form and admit of correspondingly simple solutions. In general, any one of several methods may be employed in solving the differential equations of electrical circuit theory. The methods given in the illustrative examples are most elementary, and little difficulty will be experienced in performing the great majority of the indicated operations. It is suggested that, as the student becomes proficient in solving circuit equations by elementary methods, he test out other methods of attack and in this way find the type of solution best adapted to his own mathematical background.

The Mathematical Appendix.—The mathematics of elementary circuit theory is given in appendix form, and it is suggested that the student make free use of the explanations, proofs, and tables that are presented. The Appendix is written particularly for the student who is familiar with the elementary calculus but who has not yet taken a formal course in differential equations or other advanced mathematical courses. Brief résumés of four, more or less distinct, mathematical topics are presented in accordance with the outline given below.

I. The Differential Equations of Elementary Circuit Theory.
- (*a*) Distinguishing characteristics of circuit equations.
- (*b*) Definitions of common terms connected with differential equations.
- (*c*) Conventional methods of solution.

II. Heaviside's Operational Calculus.
- (*a*) The direct operational method.
- (*b*) Heaviside's expansion theorem.
- (*c*) Reference to special devices.

III. Graeffe Method of Solving Algebraic Equations.
- (*a*) General explanation of the method.
- (*b*) Solution of third degree equations. Real and complex roots.
- (*c*) Solution of fourth degree equations. Real and complex roots.

IV. Exponential and Hyperbolic Functions.
- (*a*) Simple exponentials. Table of numerical values.
- (*b*) Characteristics of ϵ^{jax} and ϵ^{x+jy}, where $j = \sqrt{-1}$.
- (*c*) Characteristics of hyperbolic functions. Table of numerical values of sinh, cosh, and tanh.

THE MATTER OF UNITS

Much attention is given to the matter of units in the field of engineering. This follows as a direct consequence of the large number of numerical calculations that must be performed. The engineer deals with equations which specify relations between quantities, and viewed from a practical standpoint, the results obtained from these equations are worthless unless the units of the final numerical answer can be stated with certainty. The final numerical result of a given problem is very likely to be in error unless strict attention has been paid to the matter of units throughout all the numerical calculations. On account of the relatively large number of units encountered in his field, the student of electrical engineering is urged to give the subject of "units" careful and continual attention.

Common Systems of Units.—The electrical engineer who concerns himself principally with electricity in motion encounters two ordinary systems of units in his work, namely:

1. The absolute electromagnetic system (the ab- units).
2. The practical electromagnetic system (the practical units).

A third system, the absolute electrostatic, is only occasionally encountered. The particular problem at hand will determine which of the systems is the most logical to employ. Generally speaking, the absolute systems find their greatest use in the field of theoretical derivations whereas the practical system is used almost exclusively in making quantitative determinations.

Electric and magnetic units were originally defined in terms of the metric system of units. Unless the definitions of certain basic quantities are changed we are definitely confronted with the fact that the absolute system of units, involving as it does the centimeter, gram, and second, is the basic system of units in the field of electrical engineering. The practical system has evolved for well-known reasons. When only the simple electric circuit quantities are involved, the practical system is as flexible and consistently useful as the absolute system even in the field of theoretical derivations. The magnetic quantities, however, do not in general lend themselves to the practical system of units without the aid of conversion factors. This follows as a direct consequence of the fact that, even in elementary magnetic circuits, we are dealing with a "field" problem with its attendant lengths and areas.

A certain confusion exists at the present time as to what shall be called a practical magnetic unit. The question regarding the magnitude

of a practical unit of magnetomotive force, for example, is still a debatable one, and marked differences of opinion exist.[1] Until the question as to what shall constitute practical magnetic units is settled, a sound procedure is to employ the absolute electromagnetic system of units in all fundamental derivations involving magnetic quantities. Then by means of proper conversion factors these expressions can be changed into working "English" forms. Progress in the construction of a practical set of magnetic units is shown by the fact that the practical unit of magnetic flux, 10^8 maxwells, has recently been given its official name, the "weber." The weber is a most useful practical unit since the rate of change of magnetic flux in a single-turn electric circuit expressed in webers per second is numerically equal to the volts induced.

Fundamental, Primary, and Secondary Units.—Units may be classified as fundamental, primary, or secondary. Three units of a given system are selected as fundamental. In the case of the practical electromagnetic system, the ampere, the ohm, and the second are usually considered to be the fundamental units. All other units in the system are dependent, in one way or another, upon the fundamental units. The primary units are defined or derived in terms of the three fundamental units of the system; the secondary units are arbitrarily selected to facilitate numerical expression. On certain occasions a physical quantity comes to us expressed in some convenient secondary unit. A conversion to primary units is, in general, necessary before the quantity can be entered into a basically derived equation. For example, let it be supposed that numerical values of R and L are to be entered into the voltage equation given on page viii. Assume that we have R expressed in ohms and L in millihenrys. If the practical system of units is decided upon, R ohms may be entered into the equation directly since it is a primary (or fundamental) unit in that system. The millihenrys, however, must be converted to the primary unit of self-inductance, namely, the henry, before they are entered into the voltage equation. The reason is obvious: $L\,\dfrac{di}{dt}$ must represent volts, primary units, and volts are represented by L in primary units (henrys) times $\dfrac{di}{dt}$ in primary units (amperes per second).

General Equations Become Restricted in Meaning with the Introduction of Particular Numerical Values.—General physical laws may,

[1] A. E. Kennelly, "Actions on Electric and Magnetic Units," *Electrical Engineering* (A.I.E.E.), March, 1934.

of course, be expressed without regard for the units of the quantities involved. The truth of the statement expressed in the energy equation on page viii is, for example, not dependent upon any system of units. If, however, any numerical values are to be entered into the equation a systematic set of units should be employed; in any event the units must be chosen so that each of the additive terms is expressed in the same unit of energy. Obviously, the equation represents an equality between certain amounts of energy. If a condition is imposed on the general equation such that the left member represents ergs of energy, each of the three terms on the right must represent ergs of energy. In this case, the equation is no longer a general energy equation; it now represents an equality between ergs. Since the erg is the primary unit of energy in the absolute electromagnetic system of units (as well as in the general cgs system) the orderly procedure is to consider each of the quantities in terms of its ab- unit; that is, time in seconds, resistance in abohms, self-inductance in abhenrys, capacitance in abfarads, current in abamperes, and voltage in abvolts.

For the sake of further illustration, assume that we are carrying out some extended mathematical analysis that involves an energy equation of the type shown on page viii and that at a particular point in the analysis we find it convenient or necessary to employ numerical values of R, L, and C. If the values of these quantities come to us expressed in the practical system of units (ohms, henrys, and farads), the logical procedure, other things being equal, is to insert these values directly into the equation. The other quantities (E, i, and t) will of necessity at this point take on restricted meanings, for the practical system of units now applies to all the factors in the equation. E represents voltage in volts, and i represents current in amperes. If no particular secondary unit of energy is specified we continue our analysis of the problem assuming that time is expressed in seconds, the fundamental unit of the system. The unit of energy involved in our equation is now the primary unit of energy in the practical system of units, namely, the joule or watt-second. If, in the final analysis, the amount of energy involved corresponds to that ordinarily encountered in engineering practice, the final results will probably be expressed in terms of some larger secondary unit of energy, such as the kilowatt-hour.

Generality of an Expression Requires That No Units Be Specified.—It is undesirable to restrict an expression which will apply generally, to any particular system of units. For this reason, the matter of units is conscientiously avoided in the derived expressions which are found in this text. The numerical calculations are carried out in almost all

cases employing units of the practical system. Generally speaking, however, either the absolute or the practical system of units may be employed in quantitative manipulations of the expressions. It is expected that the student will become sufficiently well acquainted with the derived expressions to know which system of units is most applicable in a particular case.

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INTRODUCTION TO ELECTRIC TRANSIENTS

SECTION I

DIRECT-CURRENT TRANSIENTS

CHAPTER I

ELEMENTARY CIRCUIT CONCEPTS

Circuit Parameters.—The foundation stones of electric circuit analysis are Kirchhoff's laws and Lenz's law. Approximately one hundred years of experimentation have demonstrated the exactness with which the laws operate. In order to apply these basic laws, however, circuit parameters such as resistance, self-inductance, series capacitance, and mutual-inductance must be employed. It is the purpose of this chapter to review certain basic concepts in connection with the three most commonly used circuit parameters, namely, resistance, self-inductance, and series capacitance.

These circuit parameters, together with the applied voltage, govern the movement of electricity in simple series circuits. The R factor opposes, directly, the flow of electricity; the L factor opposes any change in the flow; and the C factor is a measure of the circuit's ability to permit flow of electricity through the circuit. Certain inherent characteristics of each of the factors may be reviewed by considering the transient response of a circuit possessing only one of these parameters. Any discussion of a purely resistive, purely inductive, or purely capacitive circuit is of course hypothetical inasmuch as such circuits are unrealizable in practice. However, by minimizing certain effects and neutralizing others, circuits that closely simulate pure R, L, or C circuits are obtainable.

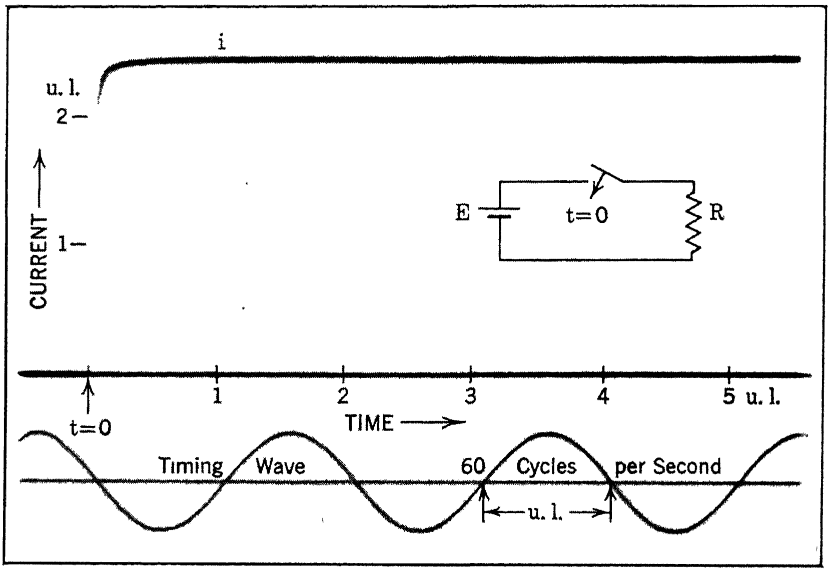
The R Circuit.—Assuming that a circuit has no inductance and unlimited series capacitance, the expression for dynamic equilibrium is the well-known relation:

$$Ri = E \tag{1}$$

Viewed from the standpoint of dynamics the statement merely asserts that the counter voltage, Ri, must equal the driving electromotive

force, E. The resistance, R, is the factor which, if multiplied by the current, establishes the expression for the *resistive* counter voltage. It is the property of an electrical circuit which directly opposes the movement of electricity and causes a non-reversible transformation of electrical energy into heat. The effect of resistance comes into play only when electricity is in motion.

Equation (1) is independent of time. Therefore, in a purely resistive circuit the current must at all times equal the driving emf divided by the



Oscillogram 1.

The sudden rise of current in a highly resistive circuit.

$E = 12$ volts. $R = 7.5$ ohms. i calibration $= 0.66$ ampere per unit length.*

* The current scale is given in terms of amperes per unit length. "u. l." is shown on the oscillogram. An explanation of the system is given on page 14.

resistance. The current will vary directly with any variation of E and inversely with any variation in R. The sudden application of a constant potential difference to a resistance branch will therefore result in the current instantly acquiring its E/R value.

Oscillogram 1 shows the variation of current in a circuit which approximates a purely resistive circuit and to which a battery potential difference is suddenly applied. That the circuit possesses a small amount of self-inductance is evident from the fact that the current does not reach its full E/R value instantly.

The L Circuit.—Assuming that an inductive circuit has no resistance and an unlimited series capacitance, the expression for dynamic equilibrium is:

$$L\frac{di}{dt} = E \tag{2}$$

The statement again asserts the fact that the counter voltage, $L\frac{di}{dt}$, must equal the driving electromotive force, E. The self-inductance, L, is the factor which, if multiplied by the rate of change of current, establishes the expression for *inductive* counter voltage. Self-inductance therefore is the property of an electric circuit by which it resists any change in the state of rest or motion of the electricity in the circuit. As such it may be considered to be electrical inertia. The effect of self-inductance comes into play only when the current changes from its previous magnitude, or in other words, when the quantity of electricity which passes a given point (or cross-sectional area) of the circuit changes its rate of movement.

The sudden application of a constant potential difference to a purely inductive circuit will result in a current which is zero at $t = 0$ (time of closing the switch), and which will continue to increase steadily with time as may be shown by the solution of equation (2):

$$\frac{di}{dt} = \frac{E}{L}$$

If E and L are constant quantities:

$$i = \frac{E}{L}t + c_1$$

Since

$$i = 0, \text{ when } t = 0: c_1 = 0$$

Therefore:

$$i = \frac{E}{L}t$$

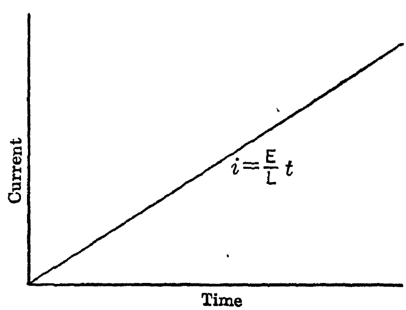
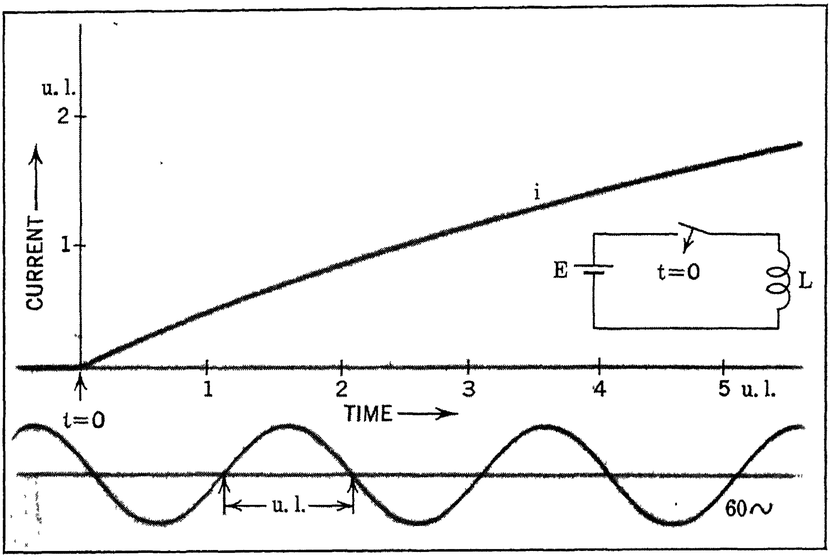


FIG. 1.—Rise of current in a purely inductive circuit.

The growth of current in an ideal L circuit with respect to time is shown in Fig. 1.

From the foregoing it is apparent that the current will increase from zero at $t = 0$ at the rate of E/L units per second, and continue to increase at this rate indefinitely because it is only by a continual increase in current that such a hypothetical circuit can be maintained in a state of equilibrium.

Oscillogram 2 shows the rise of current with respect to time in an air-core inductance coil upon the application of a constant potential difference. Since no magnetic material is employed the self-inductance is constant. The fact that the circuit also possesses resistance is in evidence inasmuch as the current does not tend to rise indefinitely. During the early part of the period, however, the rise of current conforms quite closely to its $(E/L)t$ value.



OSCILLOGRAM 2.

The gradual rise of current in a highly inductive circuit.

$E = 2.0$ volts. $L = 0.056$ henry. $R = 0.83$ ohm. i calibration $= 0.66$ amp. per u. l.

The C Circuit.—The manner in which series capacitance affects the movement of electricity is partially expressed in the following equation:

$$i = C\frac{de_c}{dt} \tag{3}$$

where $\frac{de_c}{dt}$ is the rate of change of potential difference across the plates of the condenser. For a given rate of change of voltage, C is a measure of the circuit's ability to permit current flow. It is sometimes called the permittance of the circuit. The capacitance, C, of a condenser is measured in terms of the quantity of electricity (charge) that accumu-

lates on the condenser plates per unit of applied potential difference.

Thus
$$C = \frac{Q}{E_c} \tag{4}$$

where Q is the final steady-state charge acquired by the condenser plates as a result of the plates being subjected to a potential difference E_c.

In studying the transient behavior of condenser circuits it is sometimes desirable to separate q into two components as shown below. The symbol q is used to designate the magnitude of the condenser charge at any time, t.

$$q = q_i + Q_0 \tag{5}$$

where q_i is the charge that accumulates as a result of current flow and Q_0 is the initial charge possessed by the plates.

$$q = \int_0^t idt + Q_0 \tag{6}$$

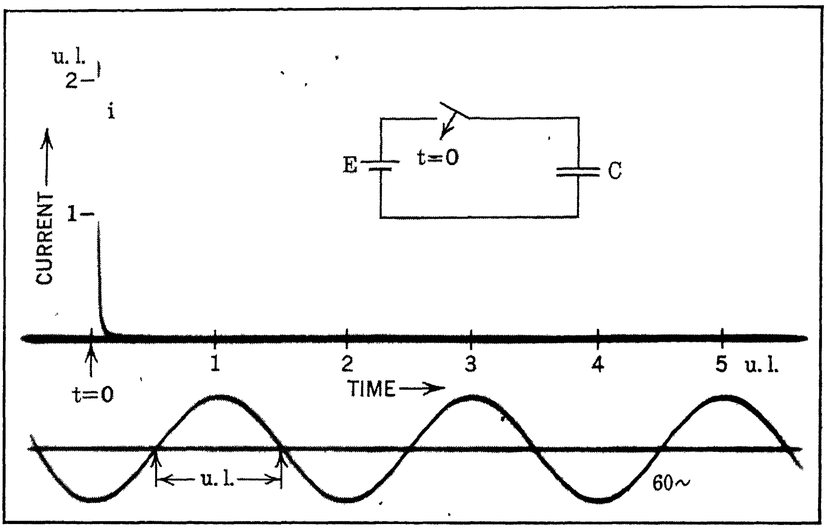
Consider the hypothetical C circuit which possesses neither resistance nor self-inductance. The application of a constant potential difference to the plates of the uncharged condenser will result in the condenser instantly acquiring a charge such that it exhibits a counter voltage that is equal and opposite (when considered around the series loop) to the applied potential difference. The expression for equilibrium is:

$$\frac{q}{C} = E \tag{7}$$

Upon the application of a potential difference the plates acquire a definite charge; the negatively charged plate acquires a definite excess of electrons and the positively charged plate becomes deficient in electrons by a corresponding amount. A movement of electricity is required for this result to be accomplished. Since the hypothetical circuit possesses neither resistance nor self-inductance, the magnitude of the current is therefore unlimited at the instant of applying the potential difference. The condenser acquires the charge necessary for equilibrium in an infinitely short period of time. Mathematical manipulations of equation (7) do not show this clearly because equation (7) is not the complete expression for dynamic equilibrium. Differentiation of (7) shows $dq/dt = i = 0$, which is true except for the very instant of closing the switch. The fact that dq/dt is equal to zero shows that whatever charge is acquired by the condenser must be acquired instantly. A graph of q with respect to time will show that dq/dt at

$t = 0$ is infinite. This indicates that the current becomes infinitely large for an infinitely short period of time, namely, at $t = 0$. This result could have been predicted from equation (3).

Oscillogram 3 shows the current variation in a circuit which approaches the ideal C circuit just described. Compared with the oscillograms previously shown, oscillogram 3 illustrates the impulsive nature of the C circuit. The R circuit and the L circuit have infinite series capacitance. There is no barrier in such circuits to the continued movement of electricity. In the R circuit the resistance limits



OSCILLOGRAM 3.
Showing the momentary current inrush to a condenser.
$E = 8$ volts. $C = 66\mu$f. $R = 0.4$ ohm.

NOTE: The electromagnetic type of galvanometer cannot record the current variation accurately in this type of circuit.

the rate of movement to a definite value, but the movement of electricity persists at that particular rate as long as a potential difference is maintained across the circuit. The self-inductance of the L circuit resists any change in the rate of movement of electricity but does not check the movement. L merely fixes the rate at which the current can change. However, a circuit in which a series condenser is placed has a limited capacitance to the movement of electricity.

Mechanical Analogies.—Frictional effects, mass, and elastance restraints govern the movement of physical bodies in the same manner that resistance, self-inductance, and capacitance govern the movement

of electricity. It is for the express purpose of further emphasizing the nature of the properties of R, L, and C that the following mechanical analogies are drawn. The conditions which are assumed are, also, physically unrealizable. It is as impossible to separate frictional and inertial effects in actual mechanical systems as it is to separate resistance and self-inductance in electrical systems.

Resistance Analogous to Damping Constant.—A mechanical analogy to a purely resistive circuit is shown in Fig. 2(*b*). The hypothetical body shown is assumed to be void of inertia and to be unrestrained along a single path of motion except for the frictional or damping effects of the surrounding medium. Upon the sudden application of a force, F, such a body would move along the path of motion with a velocity equal to F/K_d, where F is the driving force and K_d is the damping constant.

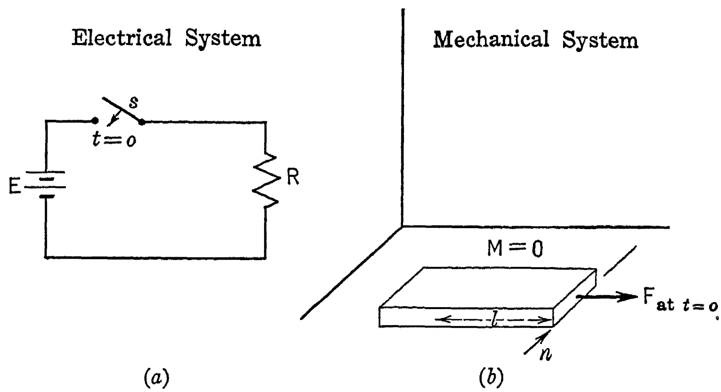


Fig. 2.—A purely resistive circuit corresponds to a physical body having no inertia.

The value of the damping constant depends upon the viscosity of the surrounding medium. Under the assumed conditions, the body will instantly acquire the velocity, F/K_d, and will maintain that velocity as long as F and K_d remain constant. The equation which states the condition for dynamic equilibrium of the system is:

$$K_d v = F \tag{8}$$

Thus $K_d v = F$ of the mechanical system is analogous to $Ri = E$ of the electrical system. The expressions are in one sense merely statements of Newton's third law of motion.

Electric Charge Analogous to Displacement.—It may be of interest to carry the analogy a bit farther in order to bring out the dimensional correspondence of the quantities in the two systems. Assume that the body in Fig. 2(*b*) is a long rod possessing the characteristics previously

mentioned. Let l be a measure of the displacement of the body; then the length of rod that passes the point n in a given length of time t_1 is:

$$l_1 = \int_0^{t_1} v dt \tag{9}$$

Assuming that v is constant and equal to V gives:

$$l_1 = Vt_1 \tag{10}$$

The amount of electricity that passes a given cross-sectional area of the electrical circuit in a given length of time t_1 is:

$$Q_1 = \int_0^{t_1} i dt \tag{11}$$

Assuming that i is constant and equal to I gives:

$$Q_1 = It_1 \tag{12}$$

Energy and Power Relations.—The energy delivered to, and expended by, the mechanical system shown in Fig. 2(*b*) is:

$$W_{me} = \int_0^{l_1} F dl = Fl_1 \tag{13}$$

The energy delivered to, and expended by, the electrical system is:

$$W_{el} = \int_0^{Q_1} E dq = EQ_1 \tag{14}$$

In both systems the entire energy delivered is transformed into heat. The rate at which mechanical energy is transformed into heat is:

$$p_{me} = F\frac{dl}{dt} = Fv \tag{15}$$

The rate at which electrical energy is transformed into heat is:

$$p_{el} = E\frac{dq}{dt} = Ei \tag{16}$$

Self-Inductance Analogous to Inertia.—A mechanical analogy to a purely inductive circuit is shown in Fig. 3(*b*). The surrounding medium is assumed to have no damping effect upon the movement of the physical body, and all frictional effects are to be neglected. The body is unre-

strained along a single path of motion except for its inertia. The expression for dynamic equilibrium of such a system is:

$$M \frac{dv}{dt} = F \tag{17}$$

where M is the mass, dv/dt is the acceleration of the body, and F is the driving or applied force. A comparison of the above expression with:

$$L \frac{di}{dt} = E$$

shows the analogous relation of L in the electrical system and M in the mechanical system. Solution of equation (17) will show that the

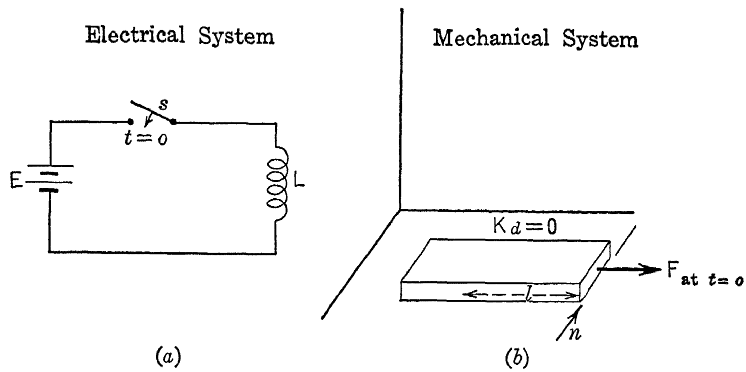


FIG. 3.—A purely inductive circuit corresponds to a physical body having no frictional effects.

velocity of the physical body is again analogous to current in the electrical circuit. And likewise it may be shown that the displacement of the body (length of rod that passes a given reference point) is analogous to the amount of electricity that passes through the circuit.

When a force is suddenly applied to the body, the tendency of the body to remain at rest is responsible for its zero initial velocity. When a voltage is suddenly applied to an inductive circuit the tendency of the circuit to remain at rest is responsible for zero initial current. t_1 seconds after the application of the force to the mechanical body, the body attains a velocity of V_1 and possesses a momentum of MV_1. t_1 seconds after the application of the potential difference to the L circuit the current builds up to I_1 and the circuit possesses an electrical momentum equal to LI_1. L is thus electrical inertia.

Energy Stored in Magnetic Field Analogous to Kinetic Energy.—The energy delivered to the mechanical body up to the time t_1 is:

$$W_{me} = \int_0^{l_1} F dl = \int_0^{V_1} M v dv = \frac{M V_1{}^2}{2} \tag{18}$$

Under the assumptions made, none of this energy has been dissipated or transformed into heat. It exists in the form of kinetic energy. The energy delivered to the electrical circuit up to the time t_1 is:

$$W_{el} = \int_0^{Q_1} E dq = \int_0^{I_1} L i di = \frac{L I_1{}^2}{2} \tag{19}$$

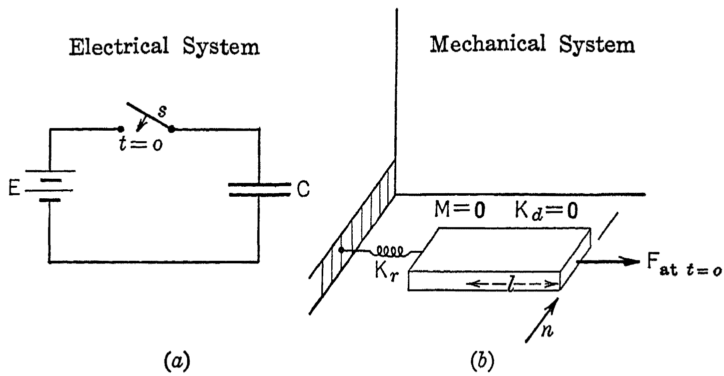


FIG. 4.—A purely capacitive circuit corresponds to a physical body which has no inertia and no frictional effects, but which is restrained in its path of motion by an ideal spring.

This energy is stored in the magnetic field and represents the kinetic energy of the electrical circuit.

Capacitance Analogous to Resilience Constant.—Fig. 4(*b*) illustrates a mechanical analogy to the ideal C circuit. The body is assumed to have no inertia and zero frictional and damping effects. The body is restrained in its path of motion by an ideal spring. The resilience constant, K_r, is the elongation (or stretch) of the spring per unit of applied force, thus:

$$K_r = \frac{l_1}{F_1} \tag{20}$$

where l_1 is the elongation corresponding to the applied force F_1. In the ideal spring, K_r is a constant.

Upon the sudden application of a steady force to such a mechanical system, the body will instantly travel along the path of motion a

certain distance, l_1. The customary manner of expressing the condition for stable equilibrium of such a system is:

$$\frac{l_1}{K_r} = F_1 \tag{21}$$

A comparison of equations (7) and (21) suggests the analogy between K_r of the mechanical system and C of the electrical system. It will be observed that, again, l in the mechanical set-up is analogous to q in the electrical set-up. l_1 is the length of rod which passes the reference point, n, as a result of the application of the force, F_1. Q_1 is the amount of electricity which passes a given point of the circuit (and hence the amount of charge that accumulates on the plates of the condenser) as a result of the application of the potential difference, E_1. Neither system shown in Fig. 4 is in stable equilibrium at $t = 0$ under the assumed conditions.

Energy Stored in Electrostatic Field Analogous to Potential Energy.—The energy delivered to the mechanical system (Fig. 4–*b*) is the summation of fdl, thus:

$$W_{me} = \int_0^{l_1} fdl \tag{22}$$

where f is the instantaneous value of the restraining force and l_1 is the final displacement.

$$f = \frac{l}{K_r}$$

$$W_{me} = \int_0^{l_1} \frac{l}{K_r}\, dl = \frac{l_1^2}{2K_r} = \frac{K_r F_1^2}{2} \tag{23}$$

In a similar manner it may be shown that the electrical energy delivered to the circuit (condenser in this case) is $CE_1^2/2$.

$$W_{el} = \int_0^{Q_1} edq \tag{24}$$

where e is the instantaneous value of the condenser voltage and Q_1 is the final charge.

$$W_{el} = \int_0^{Q_1} \frac{q}{C}\, dq = \frac{CE_1^2}{2} \tag{25}$$

In the case of the mechanical system the energy is stored in the form of potential energy as long as the body maintains its displacement, l_1, from its initial position. In the electrical system the energy is

stored in the electrostatic field and persists as such until the condenser is, by some means, allowed to discharge, hence the common analogy between potential energy and energy stored in an electrostatic field.

Electromechanical Equivalents.—In the foregoing analyses of the elementary circuits it was observed that every electrical quantity discussed had a mechanical analogue. The basic reason for this is that the laws that govern the flow of electricity in metallic circuits are of exactly the same nature as the laws that govern the dynamics of a moving body. Any three of the mechanical quantities with the corresponding three electrical quantities may be set down as fundamental and the other quantities derived in terms of the so-called fundamentals. It is customary, in the mechanical system, to employ mass, length, and time as the three fundamental quantities because of the ease with which the units may be standardized and duplicated. The dimensional qualities of all derived quantities may then be expressed in terms of mass, length, and time. The corresponding three quantities in the electrical system are self-inductance, charge, and time. It is not customary to construct a set of electrical quantities based upon these as fundamentals, as they are not easily reproducible. By international agreement, resistance and current are used instead of self-inductance and charge. It should be recognized in the study of the tabulation given below that self-inductance and charge might theoretically be considered as fundamental electrical quantities.

Fundamental Quantities

Mechanical System	Symbol	Symbol	Electrical System
Mass	M	L	Self-inductance
Length	l	q	Electrical charge
Time	t	t	Time

Derived Quantities

Mechanical System	Symbol	Dimension	Dimension	Symbol	Electrical System
Velocity	v	lt^{-1}	qt^{-1}	i	Current
Force	F	Mlt^{-2}	Lqt^{-2}	E	Voltage
Damping constant	K_d	Mt^{-1}	Lt^{-1}	R	Resistance
Resilience constant	K_r	$M^{-1}t^2$	$L^{-1}t^2$	C	Capacitance
Energy	W_{me}	Ml^2t^{-2}	Lq^2t^{-2}	W_{el}	Energy
Power	P_{me}	Ml^2t^{-3}	Lq^2t^{-3}	P_{el}	Power

Mechanical analogies are often employed as an aid in visualization, explanation, and even mathematical analysis. But the student is warned against the process of deducing his results wholly in terms of analogies. Unless he fully understands the physics of the electrical circuit under consideration, as well as the physics of the mechanical

analogy that he proposes to employ, and unless he has satisfied himself about the dimensional correspondence of the quantities with which he is dealing, analogous reasoning is likely to lead to erroneous conclusions.

Uses of Dimensional Qualities of Electrical Quantities.—The dimensions of the derived electrical quantities in the above table are given in terms of arbitrarily chosen electrical quantities. These dimensions should not be confused with the classical dimensions of electrical quantities in terms of mass, length, time, and permeability or dielectric constant.[1]

The dimensions of the electrical quantities given in terms of self-inductance, electrical charge, and time are rational and easily interpreted. Comparison of an electrical quantity with its mechanical analogue may serve as an aid in the visualization of the characteristics of the electrical quantity. Mechanical inertia, velocity, and acceleration are very much more real to most people than are the corresponding electrical quantities, self-inductance, current, and rate of change of current.

The use of dimensions in helping to check the accuracy of derived expressions is well known. Each of the additive factors in an equation must have the same dimensions, otherwise the indicated additions cannot be performed. Obviously, the dimensions of the quantities on the two sides of an equation must be the same.

The dimensions of a particular grouping of electrical quantities may point the way to a simple interpretation of their meaning. As an elementary illustration, let it be required to interpret the meaning of the factor RC. Employing either the classical dimensions or the dimensions shown in the foregoing table, the combination, RC, is found to be dimensionally equivalent to time. Another combination of circuit parameters which is sometimes encountered in circuit analysis is $\sqrt{L/C}$. Its dimension is Lt^{-1} in terms of the L–q–t system. It will also be observed that, in the L–q–t system, R has the dimension Lt^{-1}. $\sqrt{L/C}$ is thus seen to be dimensionally equivalent to electrical resistance. Both R and $\sqrt{L/C}$ "impede" the movement of electricity and come under the general classification of "impedances."

The dimensional correspondence of mechanical quantities and electrical quantities has led to a rather unique method of solving mechanical and electromechanical problems.[2] Mechanical or electromechanical

[1] Malti, "Electric Circuit Analysis," pages 40–41. Starling, "Electricity and Magnetism," page 396.

[2] C. A. Nickel, "Oscillographic Solution of Electro-Mechanical Systems," *Journal A.I.E.E.*, Vol. 44.

systems that would be extremely difficult to analyze either experimentally or mathematically are set up in terms of their electrical equivalents. Oscillographic records of currents and voltages in the equivalent electrical circuit, following some disturbance, indicate the reaction or response of the system to that disturbance. The oscillographic record thus "constitutes the plotted solution of the differential equation of the system."

EXERCISES

Scales of time, current, voltage, and power are given on many of the oscillograms in this book in terms of the length of a half cycle of the timing wave as a unit of measure. The timing wave employed throughout the text is a 60-cycle-per-second wave. The distance between zero crossings of the timing wave is called "unit length." This unit length represents $\frac{1}{120}$ of a second. Where a current scale is given as k amperes per u.l. the value of the current in amperes at any time may be determined by scaling off (vertically) the number of unit lengths at the time in question and multiplying that number of unit lengths by k.

1. Counting $t = 0$ at the point at which i starts toward its E/R value, what duration of time is depicted by Oscillogram 1? How many coulombs pass a given cross-sectional area of the R circuit during this period of time? How many calories of heat are generated during this period?

2. Assuming that the L circuit shown in connection with Oscillogram 2 possesses no resistance, find the value of L from the initial slope of the oscillographic record of i. Compare the result with the value given below the oscillogram.

3. The actual resistance of the L circuit shown in connection with Oscillogram 2 is 0.83 ohm. Does the oscillographic record show the E/R value of i? How many joules are stored in the magnetic field at $t = 0.042$ second?

4. Consider Oscillogram 3. What is the observable length of time required to charge the condenser? Why is it impossible to charge a condenser in zero time?

5. A condenser of 30-microfarad capacitance has an initial charge of 15×10^{-4} coulomb. (*a*) What is the initial condenser voltage, E_{co}? (*b*) What will be the effect of suddenly connecting the above condenser to a steady source of voltage, namely, $E = 100$ volts? Consider both polarities of E_{co}.

6. Let it be assumed that, when E and E_{co} act in the same direction around the series loop, the charge necessary for static equilibrium in 5(*b*) is transferred at a uniform rate in 0.001 second. What will be the current flow during the charging period? NOTE: The charge is not actually transferred at a uniform rate, as will be shown in Chapter II.

7. What are the dimensions of the following expressions in terms of the L–q–t system:

$$\frac{E}{R}, \quad \frac{E}{L}t, \quad \frac{Q}{C}, \quad L\frac{di}{dt}, \quad \frac{L}{R}, \quad \sqrt{LC}, \quad \text{and} \quad \frac{2L}{R}?$$

CHAPTER II

TRANSIENTS IN SERIES CIRCUITS WITH CONSTANT VOLTAGE APPLIED

Single and Double Energy Transients.—Electrical transients may be classified into two groups, namely, single energy and double energy transients. Single energy transients occur in circuits in which energy can be stored in one form only, that is, in either magnetic or electrostatic form, but not in both. Circuits containing R and L, or R and C give rise to transients belonging to this class. Double energy transients occur in circuits containing R, L, and C, in which energy may be stored in both forms. A third form of energy, heat, accompanies all movement of electricity through metallic circuits.

The present chapter deals with single and double energy transients in series circuits which are initially at rest.

THE *RL* CIRCUIT

Physical Considerations.—Let it be required to find the current in the circuit shown in Fig. 1 at the time of closing the switch and for the period immediately thereafter. E, R, and L are assumed to be constant and time is to be counted from the instant the switch is closed.

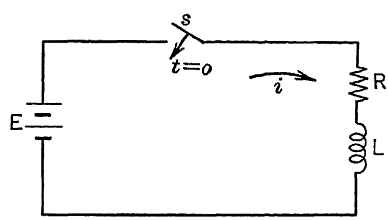


FIG. 1.—The *RL* circuit.

The expression for dynamic equilibrium in the circuit is the well-known Kirchhoff's emf law which states that, around any closed loop, the sum of the voltage drops equals the sum of the voltage rises.

$$Ri + L\frac{di}{dt} = E \tag{1}$$

The sum of the two counter-voltages must at all times equal the applied or driving emf. The self-inductance (electrical inertia) of the circuit requires that the current be zero at the time of closing the switch, namely, at $t = 0$. With $i = 0$ it is evident that the inductive counter-voltage must equal the applied emf at $t = 0$ because the resistive

drop, Ri, is at that time equal to zero. $\left(\frac{di}{dt}\right)$ at $t = 0$ is therefore equal to E/L. The steady-state value of the current, that is, the value of the current when $\frac{di}{dt} = 0$ is known to be equal to the constant value E/R. Evidently the current starts at zero, increasing at the rate of E/L units per second at $t = 0$, and continues to increase at a decreasing rate until the E/R value is attained.

The phenomenon involved during the transition period is sometimes explained by Lenz's law. Prior to $t = 0$ no magnetic flux links with the electrical circuit. As the current establishes itself in the circuit, the accompanying magnetic field necessarily changes. The change in the amount of flux linking with the circuit is responsible for a counter-voltage of self-induction, $N\frac{d\phi}{dt}$. The self-induced emf always acts in a direction such that it opposes the change of flux that is taking place. $N\frac{d\phi}{dt}$ is merely another way of expressing the counter-voltage $L\frac{di}{dt}$.

For the sake of simple analysis the current is separated into two components as shown below:

$$i = i_s + i_t \tag{2}$$

where i is the actual current.
i_s is the steady-state term.
i_t is the transient term.

It will be noted that, while one component of the total current is herein called the transient term, it is quite customary to speak of the total current during the period of transition as the transient current.

Substituting equation (2) in equation (1) gives:

$$Ri_s + L\frac{di_s}{dt} + Ri_t + L\frac{di_t}{dt} = E$$

But

$$Ri_s + L\frac{di_s}{dt} = E$$

since

$$i_s = \frac{E}{R}, \quad \text{and} \quad \frac{di_s}{dt} = 0$$

Therefore

$$Ri_t + L\frac{di_t}{dt} = 0 \tag{3}$$

The type of function that i_t must be to satisfy the latter condition may be gained by inspection. The straightforward method of separating the variables and integrating is taken up in a later paragraph. But without recourse to that method, it is possible to predict the nature of i_t. The problem is to determine the type of function which, multiplied by a constant and added to the derivative of that function, will equal zero. The function that satisfies these requirements is of the form,

$$i_t = A\epsilon^{at} \tag{4}$$

where A and a are constants. a may be determined by substituting the value of i_t in equation (3), thus:

$$(R + La)A\epsilon^{at} = 0$$

From which

$$a = -\frac{R}{L}$$

The expression for the total current then becomes:

$$i = \frac{E}{R} + A\epsilon^{-\frac{R}{L}t} \tag{5}$$

Taking into account the physical fact that $i = 0$ at $t = 0$, A is found to be equal to $-E/R$. The current may now be expressed explicitly as a function of time:

$$i = \frac{E}{R} - \frac{E}{R}\epsilon^{-\frac{R}{L}t} = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right) \tag{6}$$

A graph of current rise in an *RL* circuit is shown in Fig. 2.

It is apparent from equation (6) that if the exponent of ϵ is large owing to an elapse of time the expression $\epsilon^{-\frac{R}{L}t}$ becomes sensibly equal to zero and the current in the circuit becomes equal to E/R, its Ohm's law value. The exponential term is thus only a modifying factor, modifying Ohm's law during the starting period of the current.

Time Constant.—An examination of equation (6) will show that the ratio of R to L governs the length of time required for the current to reach a certain percentage of its E/R value. Theoretically i never reaches its E/R value but in most circuits this value is attained, for all practical purposes, in a few seconds or less. A large ratio of R to L, that is, either a large resistance or small inductance, results in the current reaching its final value more quickly than if the circuit possesses a smaller ratio of R to L. The L/R ratio has the physical dimensions

of time [1] and is known as the time constant of an inductive circuit. It offers a convenient method of comparing different RL circuits, for when $t = L/R$ the exponent of ϵ in equation (6) reduces to minus unity and the bracket term to $(1 - \epsilon^{-1})$, which is equal to $(1 - 0.368)$ or 0.632. Thus the time constant may be defined as being the length of time required for the current to rise to 63.2 per cent of its final E/R value and is:

$$t_c = \frac{L}{R}$$

It is, of course, necessary that L and R be expressed in a systematic set of units if t_c is to be in seconds.

The time constant L/R is represented in Fig. 2 as the time to the point at which the line representing the increase of current for a purely inductive circuit crosses the line representing the current curve for a purely resistive circuit. From this it follows that the time constant is the length of time required for the current to reach its maximum value if it continued to increase at its initial rate.

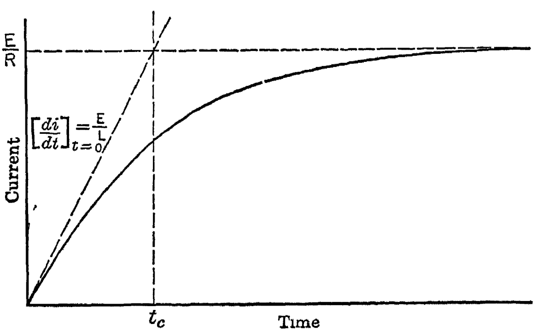


FIG. 2.—Growth of current in the RL circuit.

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right).$$

Component Voltages.—The resistive counter-voltage, Ri, is directly proportional to i and, of course, follows the same variation with respect to time as does i.

$$Ri = R \times \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right) = E\left(1 - \epsilon^{-\frac{R}{L}t}\right) \tag{7}$$

The inductive counter-voltage $L\frac{di}{dt}$, is directly proportional to the rate of change of i with respect to time.

$$L\frac{di}{dt} = L\frac{d}{dt}\left[\frac{E}{R} - \frac{E}{R}\epsilon^{-\frac{R}{L}t}\right]$$

$$= L\left[0 - \frac{E}{R} \times \left(-\frac{R}{L}\right)\epsilon^{-\frac{R}{L}t}\right] = E\epsilon^{-\frac{R}{L}t} \tag{8}$$

[1] See dimensional table, page 12.

The sum of the two counter-voltages is equal to E. Graphs of the variations with respect to time are shown in Fig. 3.

Power and Energy.—The energy delivered to the RL circuit during the transient period may be divided into two parts. A portion of the energy is transformed into heat as a result of the passage of current through the resistance. The remainder is stored in the magnetic field that links the turns of the L coil.

The rate at which energy is transformed into heat is the power absorbed by the resistance. It is:

$$Ri^2 = R \times \left[\frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)\right]^2 = \frac{E^2}{R}\left[1 - 2\epsilon^{-\frac{Rt}{L}} + \epsilon^{-\frac{2Rt}{L}}\right] \tag{9}$$

The electrical energy that is transformed into heat from $t = 0$ to $t = t_1$ may be determined by integrating equation (9) between those limits.

$$\text{Heat due to } \left[Ri^2\right]_0^{t_1} = \frac{E^2}{R}\left[t_1 + \frac{2L}{R}\epsilon^{-\frac{Rt_1}{L}} - \frac{L}{2R}\epsilon^{-\frac{2Rt_1}{L}} - \frac{3}{2}\frac{L}{R}\right] \tag{10}$$

For large values of t_1 the second and third terms within the bracket are sensibly equal to zero. The fourth is a time equal to $\frac{3}{2}$ the time constant of the circuit, and it will be observed that it subtracts directly from t_1.

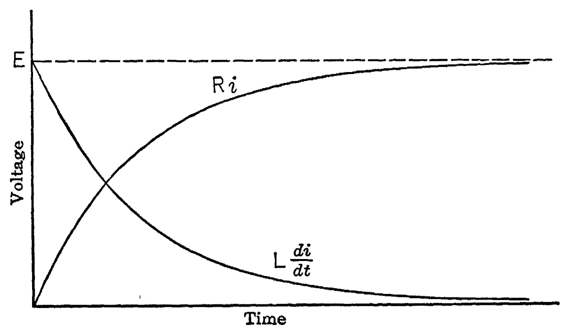


FIG. 3.—Component counter-voltages in the *RL* circuit during the period of rising current.

The rate at which energy is stored in the magnetic field is the power absorbed by the inductance. It is:

$$\left(L\frac{di}{dt}\right) \times i = E\epsilon^{-\frac{R}{L}t} \times \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)$$

$$= \frac{E^2}{R}\left(\epsilon^{-\frac{R}{L}t} - \epsilon^{-\frac{2Rt}{L}}\right) \tag{11}$$

The energy stored in the magnetic field may be obtained by integrating the above expression for power between the limits of 0 and t_1.

$$\text{Stored energy} = \frac{E^2}{R}\left[-\frac{L}{R}\epsilon^{-\frac{Rt_1}{L}} + \frac{L}{2R}\epsilon^{-\frac{2Rt_1}{L}} + \frac{L}{2R}\right] \tag{12}$$

For large values of t_1 the exponential terms become very small and the above expression takes the familiar form:

$$\frac{E^2L}{2R^2} = \frac{LI^2}{2}$$

The expression for the total power delivered to the circuit may be found by adding equations (9) and (11). It is more convenient, however, to solve for it directly in the form of Ei.

$$\text{Total power} = \frac{E^2}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right) \tag{13}$$

The variation of the total power consumption with respect to time is of the same nature as the variation of i. Fig. 4 shows the variations of the resistive and inductive components of the total power, as well as the total power.

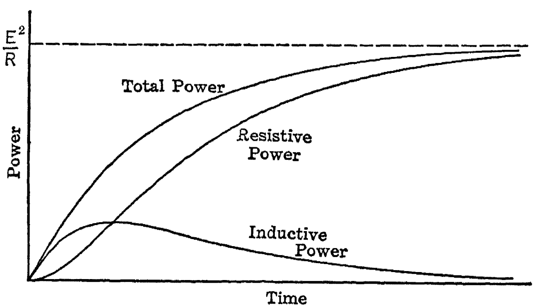


FIG. 4 —Power variations in the RL circuit.

The method that was employed in arriving at a solution to equation (1) is sometimes objectionable to the beginner because it presupposes a certain knowledge of the physical phenomena that are involved. Some prefer a more straightforward mathematical solution. Which of the two methods of attack is the better depends upon the particular problem at hand and the experience of the investigator. If the required solution can be obtained in the light of known physical laws, no better method is to be had. There are cases where the application of rigorous mathematical methods will serve to carry the problem through difficult steps to a solution that can be recognized. And again there are cases where it seems desirable to depart from the classical mathematics and employ the highly systematized procedure of the Heaviside's operational calculus. In order that these mathematical tools be available, additional mathematical solutions will be given to certain of the illustrative examples.

Solution by Conventional Method.—Given equation (1):

$$L\frac{di}{dt} + Ri = E \tag{1}$$

The solution of the above ordinary differential equation of the first order and first degree having constant coefficients may be found in the Appendix, page 267. In this particular case, the right-hand member of the equation being a constant, an easy method of solution is to separate the variables, i and t, and integrate. Transposing:

$$Ldi = (E - Ri)\,dt$$

$$\frac{di}{\left(i - \frac{E}{R}\right)} = -\frac{R}{L}\,dt$$

Integrating:

$$\log_\epsilon\left(i - \frac{E}{R}\right) = -\frac{Rt}{L} + c$$

where c is the constant of integration. It follows that:

$$i - \frac{E}{R} = c_1 \epsilon^{-\frac{Rt}{L}} \tag{14}$$

In order to determine c_1 it is necessary to know the value of i for some corresponding time, t. The fact that the RL circuit is at rest at $t = 0$ yields the necessary information, namely, $i = 0$ at $t = 0$. Substituting this boundary condition in equation (14) it is found that:

$$c_1 = -\frac{E}{R}$$

and the explicit expression for current becomes

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right) \tag{15}$$

The Operational Solution.—Letting the derivative with respect to time $\left(\frac{d}{dt}\right)$ be symbolized by p, equation (1) takes the following operational form:

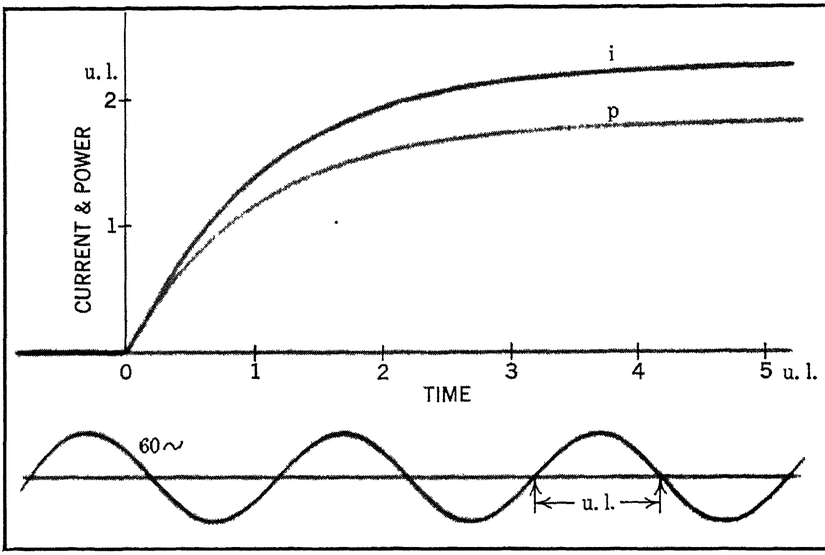
$$Lpi + Ri = E$$

or

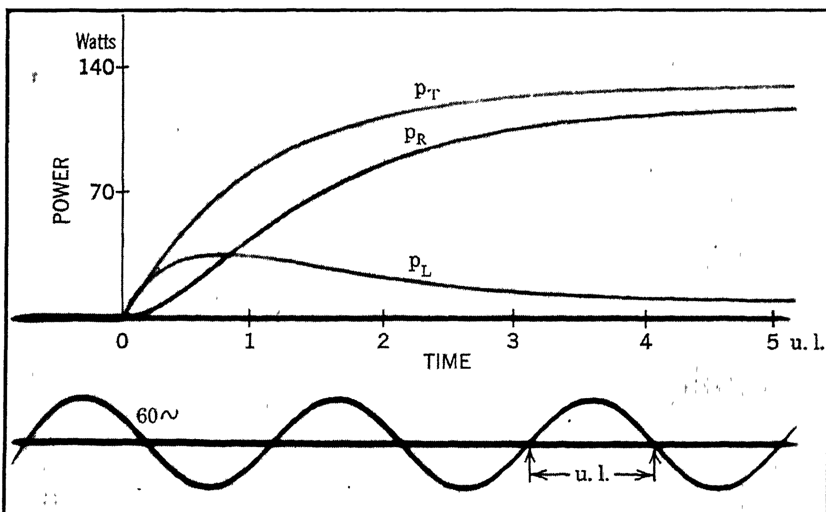
$$i(R + Lp) = E$$

from which:

$$i = \frac{E}{R + Lp} = \frac{E}{L\left(\frac{R}{L} + p\right)}$$



Oscillogram 1.
Current and power variations in the RL circuit.
$E = 29.6$ volts. $R = 6.5$ ohms. $L = 0.06$ henry.
i calibration $= 1.95$ amp. per u. l.
p calibration $= 70$ watts per u. l.



Oscillogram 2.
p_T indicates the total power delivered to the RL circuit.
p_R is the Ri^2 component and p_L is the $\left[L\dfrac{di}{dt}i + ri^2\right]$ component.
$E = 29.6$ volts. $R = 6.0$ ohms. $r = 0.5$ ohm. $L = 0.056$ henry.

On page 279 of the Appendix it is shown that the expression

$$\frac{1}{a+p}1 = \frac{1}{a}(1 - \epsilon^{-at})$$

Letting $a = R/L$:

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right) \tag{16}$$

Oscillographic Verification.—Oscillogram 1 shows the current and power variation with respect to time in an RL circuit that follows the sudden application of a battery potential to that circuit. It will be observed that both the current and the power follow the $K(1 - \epsilon^{-at})$ form that was predicted for these variations. Oscillogram 2 is an attempt to illustrate, experimentally, the rate at which energy is stored in the magnetic field of the $(R + r)L$ circuit shown in Fig. 5. Simultaneous variations of total power, resistive power, and inductive power are shown. The variation marked p_L includes the ri^2 power of the inductance coil. The oscillogram is of qualitative interest only since the potential coil of the watt-galvanometer, when shunted across R, and $(r + L)$ disturbs the original $(R + r)L$ circuit. The manner in which the Ri^2 component and the $\left[L\frac{di}{dt}i + ri^2\right]$ component combine to form the total power is also shown.

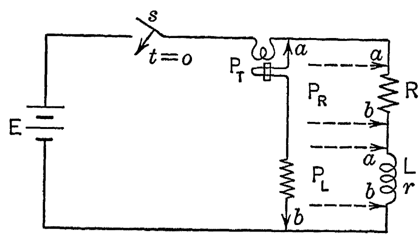


FIG. 5.—Circuit arrangement employed in obtaining the power variations shown in Oscillogram 2. With the potential element of the watt-galvanometer connected as shown the total power delivered to the $(R + r)L$ branch plus the power consumed by the potential element is recorded by the galvanometer.

THE *RC* CIRCUIT

Physical Considerations.—Let it be required to find the current in the circuit shown in Fig. 6 at the time of closing the switch and for the period immediately thereafter. E, R, and C are assumed to be constant. Let s be closed at $t = 0$, and assume that the initial charge on the condenser plates is equal to Q_0. Q_0 is considered positive if the Q_0/C voltage is a positive counter-voltage, i.e., acts in the opposite direction to E around the series loop.

The two component voltages that will be operative in counterbalancing the applied voltage are the Ri drop and the back voltage of the condenser, q/C. Since the circuit is assumed to possess zero self-inductance the initial current cannot be predicted to be of zero value. The final or steady-state value of the current is, of course, equal to zero.

Between the time of closing the switch and the time at which the current reaches its steady-state value a current in the form of a transient flows in the circuit to supply the condenser with a charge Q_s. Q_s, the final condenser charge, is of a magnitude and polarity such that the Q_s, C voltage is equal to and opposes E around the series loop. The difference between Q_s and Q_0 is the amount of charge that must be transferred from the battery to the condenser. If $Q_0/C = E$, no transfer is necessary and no transient current flows.

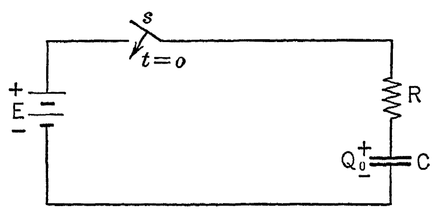


FIG. 6.—The RC circuit.

The expression for dynamic equilibrium is

$$Ri + \frac{q}{C} = E \tag{17}$$

where q is the instantaneous value of the charge on the condenser plates. In equation form:

$$q = \int_0^t i\,dt + Q_0 \tag{18}$$

Equation (17) may therefore be written:

$$Ri + \frac{\int_0^t i\,dt}{C} = E - \frac{Q_0}{C} \tag{19}$$

Since, at $t = 0$, $\int_0^t i\,dt$ is zero:

$$I_0 = \frac{E - \dfrac{Q_0}{C}}{R} \text{ (Initial current)} \tag{20}$$

When $Q_0 = 0$ the initial current is equal to E/R. The initial current may be opposite in direction to a positive battery current if $\dfrac{Q_0}{C} > E$.

The general nature of the current variation may be determined by differentiating equation (19), thus:

$$R\frac{di}{dt} + \frac{i}{C} = 0 \tag{21}$$

It will be observed that i in the RC circuit is of the same nature as i_t in the RL circuit, namely, a simple exponential variation.

$$i = A\epsilon^{at}$$

where A and a differ from the constants determined in the RL circuit analysis.

It is quite evident that A will be the initial current, $\dfrac{E - (Q_0/C)}{R}$, and that a will be some negative number. a may be determined by substituting the expression for i in equation (21). Substituting $A\epsilon^{at}$ for i in equation (21) yields:

$$RaA\epsilon^{at} + \frac{A\epsilon^{at}}{C} = 0$$

from which:

$$\left(Ra + \frac{1}{C}\right)A\epsilon^{at} = 0$$

or

$$Ra + \frac{1}{C} = 0$$

and

$$a = -\frac{1}{RC}$$

The expression for current therefore is:

$$i = \left[\frac{E - \dfrac{Q_0}{C}}{R}\right]\epsilon^{-\frac{t}{RC}} = I_0\epsilon^{-\frac{t}{RC}} \tag{22}$$

The variation in current with respect to time being known, the variations in charge, component voltages, and power are at once obtainable.

Time Constant.—The time constant of the RC circuit is the length of time required to make the exponent of ϵ minus unity, namely,

$$t_c = RC$$

When this obtains:

$$i = I_0\epsilon^{-1} = I_0\frac{1}{\epsilon} = 0.368\ I_0$$

Thus, the time constant is defined as the length of time required for the current to decrease from its initial value to 36.8 per cent of that value. In the ordinary RC circuit the time constant is a fraction of a second. At $t = RC$ the condenser charge has reached 63.2 per cent of its final value.

The effect of increasing the series resistance is to reduce the initial or maximum value of charging current, and at the same time increase the time required for the condenser to reach full charge. The area under the charging current curve must in each case be the same since this area is the product of current and time, which is numerically equal to the charge in coulombs if the quantities involved are expressed in the practical system of units. Since Q_s must be the same after the

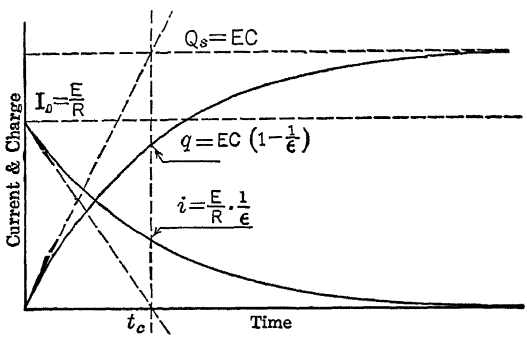


FIG. 7.—Graphical representation of the instantaneous values of charging current and the instantaneous values of condenser charge during the charging period.

charging process is completed no matter what the resistance of the circuit may be, the areas must be the same for any given transfer of charge.

Condenser Charge.—Assuming that the initial charge on the condenser is zero, the total charge on the plates at any time is:

$$q = \int_0^t i dt = \int_0^t \frac{E}{R} \epsilon^{-\frac{t}{RC}} dt$$

or

$$q = EC\left(1 - \epsilon^{-\frac{t}{RC}}\right) \tag{23}$$

A graph of the variation is shown in Fig. 7.

Component Voltages.—The voltage drop across the resistance is:

$$Ri = R \times \frac{E}{R} \epsilon^{-\frac{t}{RC}} = E\epsilon^{-\frac{t}{RC}} \tag{24}$$

The back voltage of the condenser is:

$$e_c = \frac{q}{C} = \frac{EC}{C}\left(1 - \epsilon^{-\frac{t}{RC}}\right) = E\left(1 - \epsilon^{-\frac{t}{RC}}\right) \qquad (25)$$

It will be observed that the sum of the counter-voltages is equal to the applied voltage E. Graphs of these variations are shown in Fig. 8.

Energy and Power.—The energy delivered to the *RC* circuit may be divided into two parts. The flow of current through the resistance of the circuit results in a transformation of electrical energy into heat energy, and as such it is dissipated. The remainder of the energy delivered to the circuit is stored in the electrostatic field that is caused by the establishment of the potential difference between the condenser plates.

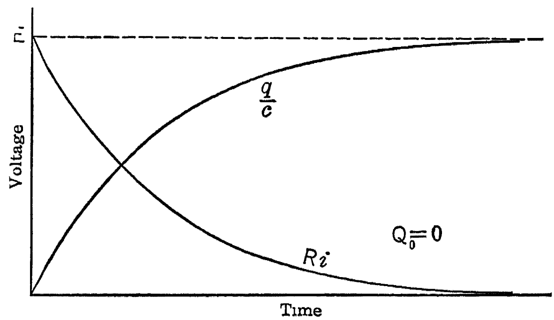


FIG. 8.—Component voltages in the *RC* circuit during the charging period.

The rate at which energy is dissipated in the resistance is:

$$Ri^2 = R \times \left[\frac{E}{R}\epsilon^{-\frac{t}{RC}}\right]^2 = \frac{E^2}{R}\epsilon^{-\frac{2t}{RC}} \qquad (26)$$

It is assumed that $Q_0 = 0$ throughout the present discussion. The rate at which energy is stored in the electrostatic field, that is, the power delivered to the condenser, is:

$$e_c i = E\left(1 - \epsilon^{-\frac{t}{RC}}\right) \times \frac{E}{R}\epsilon^{-\frac{t}{RC}} = \frac{E^2}{R}\left(\epsilon^{-\frac{t}{RC}} - \epsilon^{-\frac{2t}{RC}}\right) \qquad (27)$$

An expression for the total power delivered to the circuit, external to the battery, is:

$$Ei = \frac{E^2}{R}\epsilon^{-\frac{t}{RC}} \qquad (28)$$

Graphs of the above power variations are shown in Fig. 9. It will be observed that the two component powers added together equal the total power.

In charging a condenser through a resistance from a constant potential source, one half of the total energy delivered to the circuit

is lost in the resistance and the other half is stored in the electrostatic field. The efficiency of the charging process therefore cannot exceed 50 per cent. This statement is easily proved by substituting the expression for charging current in the energy equation, thus:

$$\int_0^\infty Eidt = \int_0^\infty i^2Rdt + \int_0^\infty \frac{q}{C}idt$$

$$E\int_0^\infty \frac{E}{R}\epsilon^{-\frac{t}{RC}}dt = \int_0^\infty \frac{E^2}{R}\epsilon^{-\frac{2t}{RC}}dt + \frac{1}{C}\int_0^Q qdq$$

Integration yields:

$$E\left[\frac{E}{R}(-RC)\epsilon^{-\frac{t}{RC}}\right]_0^\infty = \left[\frac{E^2}{R}\left(-\frac{RC}{2}\right)\epsilon^{-\frac{2t}{RC}}\right]_0^\infty + \frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q$$

or:

$$E^2C = \frac{E^2C}{2} + \frac{E^2C}{2} \tag{29}$$

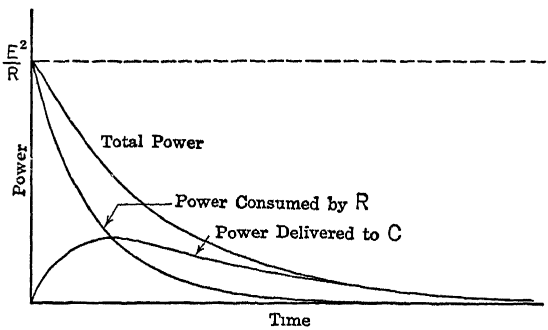


FIG. 9.—Power variations in the RC circuit during the charging period.

The left-hand term represents the total energy supplied by the circuit; the right-hand terms represent the energy lost in the resistance as heat and the energy stored in the condenser. Since the two right-hand terms are equal in magnitude, as much energy is lost during charging as is actually stored.

Mathematical Analysis.—Given equation (17):

$$Ri + \frac{q}{C} = E$$

A straightforward solution of this equation may be had in terms of either i or q. The solution will be given in terms of i. Assuming that $Q_0 = 0$:

$$Ri + \frac{\int_0^t idt}{C} = E \tag{17a}$$

Differentiating and rearranging equation (17a):

$$\frac{di}{dt} + \frac{i}{RC} = 0 \tag{30}$$

Separating variables and integrating:

$$\frac{di}{i} = -\frac{dt}{RC}$$

$$\log_\epsilon i = -\frac{t}{RC} + c$$

which reduces to:

$$i = c_1 \epsilon^{-\frac{t}{RC}} \tag{31}$$

To find the value of the constant of integration, c_1, the expression for i may be substituted in equation (17) with the boundary conditions pertaining to condenser charge imposed upon that equation. For the case under consideration $q = 0$ at $t = 0$.

$$c_1 = \frac{E}{R}$$

Therefore:

$$i = \frac{E}{R}\,\epsilon^{-\frac{t}{RC}} \tag{32}$$

Operational Solution.—The operational form of equation (17) is:

$$Ri + \frac{i}{pC} = E \tag{33}$$

since $q = \int_0^t i\,dt$ and $\int_0^t \cdot dt$ is symbolized by $\frac{1}{p}$. The explicit expression for i is:

$$i = \frac{E}{R + \frac{1}{pC}} = \frac{E}{R} \cdot \frac{1}{1 + \frac{1}{RCp}} \tag{34}$$

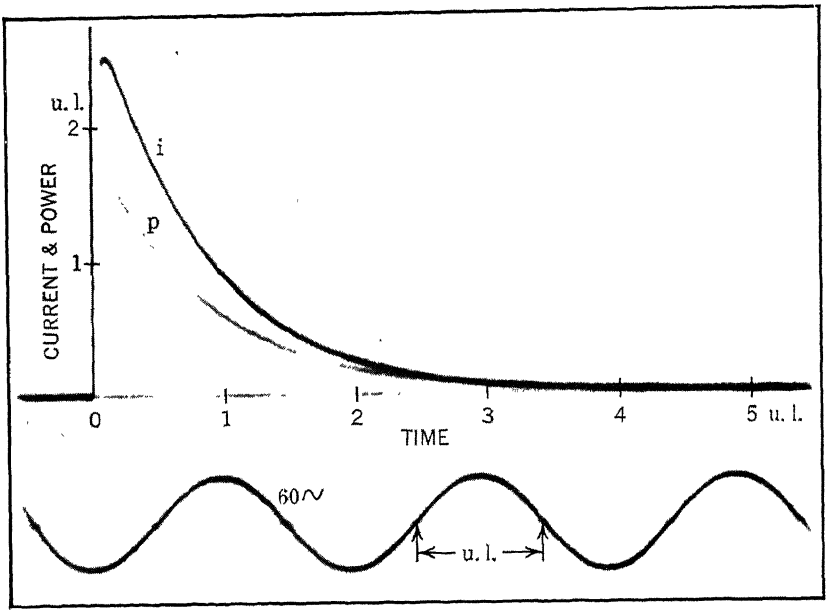
It is shown on page 279 that:

$$\frac{1}{1 + \frac{a}{p}} 1 = \epsilon^{-at}$$

Hence, letting $a = \frac{1}{RC}$:

$$i = \frac{E}{R} \cdot \epsilon^{-\frac{t}{RC}} \tag{35}$$

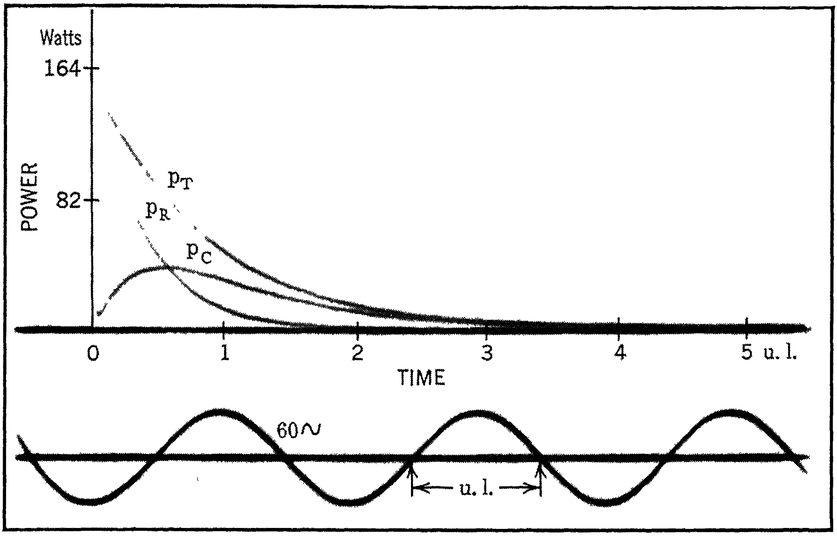
Oscillographic Verification.—Oscillogram 3 shows the variations in current and power with respect to time in the *RC* circuit following the sudden application of a constant battery potential to that circuit. Both variations are of approximately the $K\epsilon^{-at}$ form. The self-inductance of the current coil of the watt-galvanometer plus the inherent self-inductance of R prevents the current from reaching its E/R value



OSCILLOGRAM 3.
Current and power in the *RC* circuit.
$E = 48$ volts. $R = 15$ ohms. $C = 460\mu$f.
i calibration $= 1.16$ amp. per u. l.
p calibration $= 82$ watts per u. l.

at $t = 0$. Closer approaches to the $K\epsilon^{-at}$ form of variation may be obtained by eliminating more of the self-inductance.

The total power and the two component powers are shown in Oscillogram 4. The p_c variation is an attempt to show, experimentally, the rate at which energy is delivered to the condenser. The circuit disturbance due to the insertion of the potential element of the watt-galvanometer causes a small discrepancy in the component powers. However, the general manner in which the component powers combine to form the total is well illustrated by the oscillogram.



OSCILLOGRAM 4.

p_T indicates the total power delivered to the *RC* circuit.

p_R is the Ri^2 component and p_C is the $\left[\frac{q}{C} \times i\right]$ component.

$E = 48$ volts. $R = 15$ ohms. $C = 460\mu$f.

THE *RLC* CIRCUIT

Physical Considerations.—A physical analysis of the behavior of a circuit containing resistance, self-inductance, and series capacitance will now be made with a view toward predicting the nature of the current variation that follows the sudden application of a constant potential difference to such a circuit.

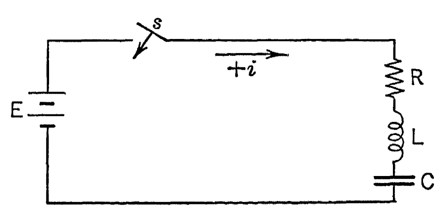


FIG. 10.—The *RLC* circuit.

Let the switch s, Fig. 10, be closed at $t = 0$, and assume that the initial condenser charge is zero. Also, let it be assumed that the circuit parameters R, L, and C are constant. The lumped R is assumed to include all the resistance, the lumped L all the self-inductance, and the lumped C the entire series capacitance of the circuit.

The self-inductance of the circuit requires that the current be zero at $t = 0$. The series capacitance requires that the final or steady-

state value of the current also be zero. The latter is accomplished by the condenser acquiring a charge, Q_s, such that the back voltage of the condenser, Q_s/C, is equal to E. The current that flows in the circuit immediately after the closing of the switch will thus be transient, lasting for the period required by the condenser to acquire its steady-state charge Q_s. Consider the voltage equation for dynamic equilibrium:

$$Ri + L\frac{di}{dt} + \frac{q}{C} = E \tag{36}$$

The nature of each counter-voltage is governed by the given circuit parameter. The Ri component is at any time directly proportional to the current that is flowing in the circuit. The $L\frac{di}{dt}$ component is directly proportional to the rate of change of the current, being a positive counter-voltage for positive rates of change of current and a negative counter-voltage for negative rates of change of current. The q/C term depends upon the charge that has accumulated on the condenser plates as a result of the current flow, assuming that $Q_0 = 0$.

A current will begin to flow immediately after the switch is closed. It will start to increase at the rate of E/L units per second; and the fact that the current must return to an ultimate zero value requires that it rise to a maximum and then decrease to zero. In so doing $\frac{di}{dt}$ begins with positive values, passes through zero value, and then changes to negative values. It is negative during the period in which the current decreases from its maximum value. At the point of maximum current $\frac{di}{dt}$ is equal to zero, and

$$Ri + \frac{q}{C} = E$$

At this particular moment the impressed emf is counterbalanced by the Ri drop and the condenser voltage. Obviously the condenser voltage, $\frac{\int_0^t i\,dt}{C}$, continues to increase as long as a positive current flows. With a decrease in the magnitude of the current after the maximum value has been reached the $-L\frac{di}{dt}$ counter-voltage becomes a driving voltage and the behavior of the circuit is largely dependent upon its value relative to the value of the Ri counter-voltage. If, during this

period, the Ri term remains greater than the $-L\dfrac{di}{dt}$ term, the back condenser voltage q/C must always be less than E. That is:

$$\frac{q}{C} = E - \left(Ri - L\frac{di}{dt}\right) \tag{37}$$

$$\frac{q}{C} < E \text{ for } Ri > -L\frac{di}{dt}$$

Under these conditions the condenser charge builds up gradually to its final value, Q_s. For a given series capacitance, a large ratio of R to L

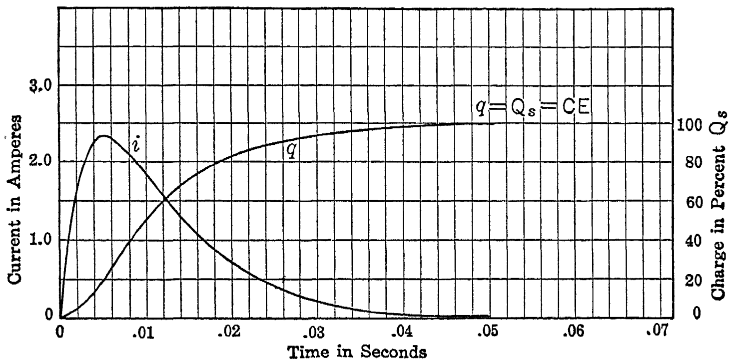


FIG. 11.—Current and charge variations in the *RLC* circuit for the condition where the Ri voltage is always greater than the $-L\dfrac{di}{dt}$ voltage. In this case the q/C voltage never exceeds the applied voltage. In the particular case shown $E = 85$ volts, $R = 27.5$ ohms, $L = 0.07$ henry, and $C = 450\ \mu$f.

tends to produce this phenomenon. The general trends of the current and charge variations would be as shown in Fig. 11.

Since $i = \dfrac{dq}{dt}$, it is apparent that the slope of the (q) charge curve indicates the magnitude of the current. Maximum current occurs at an inflection point on the charge curve, namely, when $\dfrac{d^2q}{dt^2} = 0$.

If the ratio of R to L is small the $-L\dfrac{di}{dt}$ term may become greater than the Ri term, in which case the phenomenon is somewhat changed. From equation (37) it is evident that q/C becomes larger than E for the period in which $-L\dfrac{di}{dt}$ is greater than Ri. At the time q reaches its

maximum value the current is at zero value. But the circuit will not settle into static equilibrium with $\frac{q}{C} > E$, and therefore a negative current flows in order to relieve the condenser of its excess charge. Thus an oscillatory current may be predicted. Fig. 12 shows the general nature of the expected variations in current and condenser charge under the conditions just described. The decreasing amplitudes of the successive current maxima are caused by the damping effect of the resistance in the circuit.

The possible oscillatory nature of the current might have been

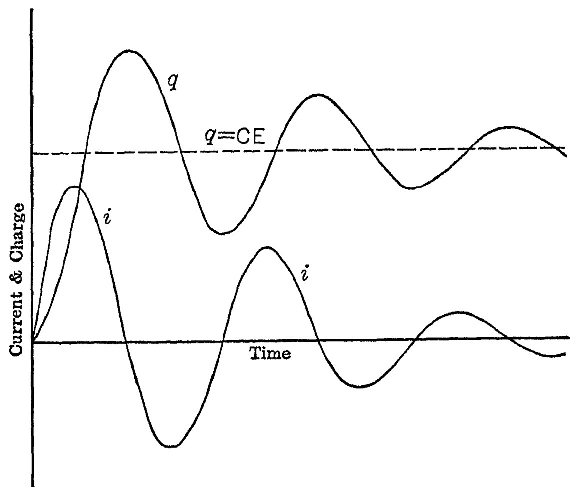


Fig. 12.—Current and charge variations in the RLC circuit for the condition where the $-L\frac{di}{dt}$ voltage becomes greater than the Ri voltage. In this case q/C is at times greater than the applied voltage, E.

predicted by considering the energy transfers that take place. The energy returned to the circuit from the magnetic field during the time that the current decreases from its positive maximum to zero furnishes the impetus for the current to take on negative values. The energy thus returned to the circuit, minus that portion which is dissipated in the resistance, is stored in the electrostatic field. If at any time during this process the energy stored in the electrostatic field exceeds $CE^2/2$, a deliverance of that excess energy to the circuit must take place before static equilibrium obtains. The interchange of energy between the electromagnetic and electrostatic fields thus causes the oscillatory current. The fact that energy is continually being dissipated in the

resistance of the circuit gradually lessens the magnitudes of the successive energy transfers.

Mathematical Analysis.—The equation to be solved is:

$$Ri + L\frac{di}{dt} + \frac{q}{C} = E \tag{38}$$

The solution may be carried through in terms of either dependent variable, i or q. Differentiating equation (38):

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

Rearranged,

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{39}$$

Equation (39) is a first degree, second order differential equation in terms of i and its derivatives.[2] Since the coefficients are constant and the right-hand member is zero a relatively simple solution is possible. It is evident that the expression for i which will satisfy this equation must be a function of time and must be of such form that the sum of the second derivative of the function plus the first derivative multiplied by R/L plus the function itself multiplied by $1/(LC)$ must be zero. By trying promising expedients it is found that if the current is of the form:

$$i = A\epsilon^{\alpha t} \tag{40}$$

equation (39) will be satisfied, since:

$$\frac{di}{dt} = \alpha A\epsilon^{\alpha t} \quad \text{and} \quad \frac{d^2i}{dt^2} = \alpha^2 A\epsilon^{\alpha t}$$

Substituting in equation (39):

$$\alpha^2 A\epsilon^{\alpha t} + \frac{R}{L}\alpha A\epsilon^{\alpha t} + \frac{1}{LC}A\epsilon^{\alpha t} = 0$$

or

$$A\epsilon^{\alpha t}\left(\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC}\right) = 0 \tag{41}$$

Equation (41) must be identically equal to zero irrespective of the value of t if expression (40) is to satisfy equation (39). Equation (41) may be solved by setting either factor equal to zero, but since no useful solution

[2] A general type of solution of a first degree, second order differential equation may be found on page 272.

would result by equating $A = 0$, the parenthesis is set equal to zero, thus:

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$$

Solving the quadratic:

$$\alpha = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

The two roots are:

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\alpha_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

In abbreviated form:

$$\alpha_1 = -a + b$$

$$\alpha_2 = -a - b$$

The complete expression for current is:

$$i = A_1\epsilon^{(-a+b)t} + A_2\epsilon^{(-a-b)t} \tag{42}$$

or

$$i = \epsilon^{-at}(A_1\epsilon^{+bt} + A_2\epsilon^{-bt}) \tag{43}$$

The values of A_1 and A_2 must come from known physical facts. The conditions assumed are:

$$i = 0 \quad \text{at} \quad t = 0$$

and

$$q = 0 \quad \text{at} \quad t = 0$$

Cases where i and q have values other than zero at $t = 0$ are considered in Chapter III.

The initial conditions imposed upon equations (38) and (43) yield, respectively:

$$\left[\frac{di}{dt}\right]_{t=0} = \frac{E}{L}$$

and

$$A_2 = -A_1$$

It follows that:

$$\left[\frac{di}{dt}\right]_{(t=0)} = A_1b - A_2b$$

Simultaneous solution for A_1 and A_2 yields:

$$A_1 = \frac{E}{2bL} \quad \text{and} \quad A_2 = -\frac{E}{2bL}$$

Therefore

$$i = \frac{E}{2bL}\,\epsilon^{-at}(\epsilon^{bt} - \epsilon^{-bt}) \tag{44}$$

The quantity a is always real, thereby causing a continual decrease in the current with increase in time. The quantity $(\epsilon^{bt} - \epsilon^{-bt})$ which determines the general mode of variation is governed to some extent by the quantity b. b may be either real or imaginary, accordingly as $\frac{R^2}{4L^2}$ is greater or less than $\frac{1}{LC}$. The transition from real to imaginary takes place when

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

Case I: $$\frac{R^2}{4L^2} > \frac{1}{LC} \quad \text{or} \quad R^2 > \frac{4L}{C}$$

Equation (44) reduces at once to:

$$i = \frac{E}{bL}\,\epsilon^{-at}\left(\frac{\epsilon^{bt} - \epsilon^{-bt}}{2}\right)$$

or

$$i = \frac{E}{bL}\,\epsilon^{-at} \sinh bt \tag{45}$$

since

$$\frac{\epsilon^{bt} - \epsilon^{-bt}}{2} = \sinh bt$$

The derivative of the current with respect to time is:

$$\frac{di}{dt} = \frac{E\epsilon^{-at}}{bL}(b \cosh bt - a \sinh bt) \tag{46}$$

Since the maximum value of the current occurs when:

$$\left[\frac{di}{dt}\right] = 0$$

it follows that:

$$t \text{ (at point of max. } i) = \frac{1}{b}\tanh^{-1}\left(\frac{b}{a}\right) \tag{47}$$

The current graph of Fig. 11 represents the plotted solution of equation (45) for a particular set of circuit parameters. The current variation is an exponentially damped hyperbolic sine. The charge varies from zero to EC in accordance with the functions of time shown in equation (61), page 42.

Case II: $$\frac{R^2}{4L^2} < \frac{1}{LC} \quad \text{or} \quad R^2 < \frac{4L}{C}$$

Since b is now an imaginary quantity certain transformations are desirable.

Rearranged:

$$b = j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

where

$$j = \sqrt{-1}$$

Let:

$$\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \beta, \text{ a real quantity.}$$

Then

$$b = j\beta$$

and

$$i = \frac{E}{\beta L}\epsilon^{-\alpha t}\left(\frac{\epsilon^{j\beta t} - \epsilon^{-j\beta t}}{2j}\right) \tag{48}$$

Since:

$$\frac{\epsilon^{+j\beta t} - \epsilon^{-j\beta t}}{2j} = \sin \beta t$$

$$i = \frac{E}{\beta L}\epsilon^{-\alpha t}\sin \beta t \tag{49}$$

The above variation in current is that of a damped sine wave, namely, a sine wave having a maximum value of $\frac{E}{\beta L}$, multiplied by the damping factor $\epsilon^{-\alpha t}$. The resulting variation, if graphed, is an oscillatory curve bounded by the envelopes, plus and minus $\frac{E}{\beta L}\epsilon^{-\alpha t}$. Reckoning the duration of a cycle between the zero points of the curve as the curve starts toward a positive maximum, it is evident that the variation has a constant period equal to $2\pi/\beta$ and an angular velocity of β. The two halves of any loop of a damped sine wave are not symmetrical. The maximum value occurs before the quarter-cycle point. A graphical representation of a damped sine wave is shown in Fig. 13.

Frequency of Oscillation.—Since the angular velocity is β:

$$2\pi f = \beta$$

or:

$$2\pi f = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

and:

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{50}$$

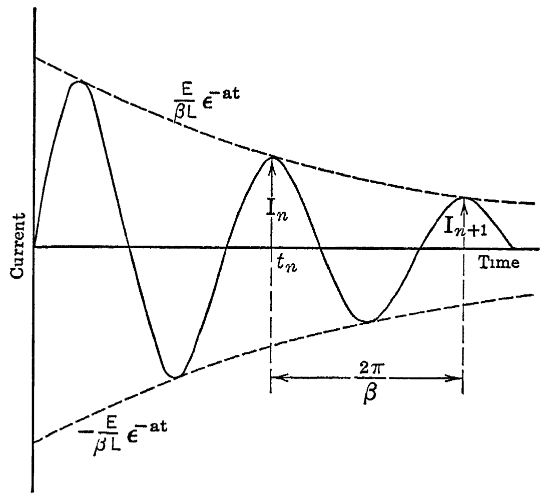


FIG. 13.—An exponentially damped sine wave variation.

In circuits having such a low resistance that $\frac{R^2}{4L^2}$ is negligible, expression (50) takes the form:

$$f = \frac{1}{2\pi\sqrt{LC}} \tag{50a}$$

which is called the "undamped" frequency.

Decrement of Damped Sine Wave Variation.—

$$\frac{di}{dt} = \frac{E}{\beta L}\epsilon^{-at}\,[\beta \cos \beta t - a \sin \beta t]$$

$$t_{(\text{1st max. } i)} = \frac{1}{\beta}\tan^{-1}\frac{\beta}{a} = \frac{\sigma}{\beta}$$

where

$$\sigma = \tan^{-1}\frac{\beta}{a}$$

Any positive maximum current may be expressed:

$$I_n = \frac{E}{\beta L} \epsilon^{-at_n} \sin \sigma$$

or

$$I_n = \frac{E\sqrt{LC}}{L} \epsilon^{-at_n} \qquad (51)$$

where t_n is the time at which the nth maximum of the current occurs.

The succeeding positive maximum current is:

$$I_{(n+1)} = \frac{E\sqrt{LC}}{L} \epsilon^{-a\left(t_n + \frac{2\pi}{\beta}\right)} \qquad (52)$$

From these two expressions the numerical decrement, which is herein defined as the difference between any two current peaks separated by a complete cycle divided by the larger of the two, may be calculated:

$$\text{Numerical decrement} = \frac{I_n - I_{n+1}}{I_n} = \left(1 - \epsilon^{-\frac{R}{2Lf}}\right) \qquad (53)$$

A more convenient method, from a theoretical point of view, of measuring the rapidity with which the oscillations die away is known as the "logarithmic decrement." It is the logarithm of the ratio of two consecutive positive current maxima:

$$\text{Logarithmic decrement} = \log_\epsilon \frac{I_n}{I_{n+1}} = \frac{R}{2Lf} \qquad (54)$$

Case III: $$\frac{R^2}{4L^2} = \frac{1}{LC} \quad \text{or} \quad R^2 = \frac{4L}{C}$$

In this case, b is equal to zero and $\alpha_1 = \alpha_2 = -a$. With $b = 0$, equation (44) takes an indeterminate form:

$$i = \frac{E}{2L} \epsilon^{-at}\left(\frac{\epsilon^{bt} - \epsilon^{-bt}}{b}\right) = \frac{E}{2L} \epsilon^{-at} \times \frac{0}{0} \qquad (55)$$

It may be evaluated by differentiation, thus:

$$\left[\frac{\frac{d}{db}(\epsilon^{bt} - \epsilon^{-bt})}{\frac{d}{db} b}\right] = \left[\frac{t(\epsilon^{bt} + \epsilon^{-bt})}{1}\right]_{b=0} = 2t$$

Substituting this value for the parenthesis in equation (55) gives:

$$i = \frac{E}{2L}\epsilon^{-at}\cdot 2t = \frac{E}{L}\epsilon^{-at}t \tag{56}$$

which is the equation for the current when R has the critical value, namely:

$$R = \sqrt{\frac{4L}{C}}$$

The plot of the current as given by equation (56) is a single surge similar to that of Case I, it being the critically damped case.

The time at which the maximum current occurs may be obtained by setting the first derivative of the current equal to zero and solving for t, thus:

$$\frac{d}{dt}\left(\frac{E}{L}\epsilon^{-at}t\right) = \frac{E}{L}\left[\epsilon^{-at} - at\epsilon^{-at}\right] = 0$$

from which:

$$\text{Time of max. } i = \frac{1}{a} = \frac{2L}{R}$$

The maximum value of the current is:

$$i_{\text{max.}} = 0.736\frac{E}{R}$$

The Operational Solution.—The expression for dynamic equilibrium in the *RLC* circuit with zero initial charge on the condenser takes the following operational form:

$$Ri + Lpi + \frac{i}{pC} = E \tag{57}$$

If the condenser has an initial charge, Q_0, the corresponding condenser voltage Q_0/C must be written into equation (57) because

$$\frac{i}{p} = \int_0^t idt$$

As such it is only the charge which the condenser acquires as a result of current flow.

$$i = \frac{E}{R + Lp + \dfrac{1}{pC}}$$

or

$$i = \frac{Ep}{L\left(p^2 + \dfrac{R}{L}p + \dfrac{1}{LC}\right)} \tag{58}$$

It has been shown that

$$\frac{p}{p+\alpha}1 = \epsilon^{-\alpha t}$$

If it is possible to arrange equation (58) into component parts of the above form, the solution for i may be effected directly. Adopting the same nomenclature as was used in the conventional solution:

$$i = \frac{Ep}{L}\left\{\frac{1}{(p-\alpha_1)(p-\alpha_2)}\right\}1 \tag{59}$$

By partial fractions equation (59) is reducible to:

$$i = \frac{E}{2bL}\left\{\frac{p}{p-\alpha_1}1 - \frac{p}{p-\alpha_2}1\right\} \tag{59a}$$

Substituting for the value of $\frac{p}{p-\alpha_1}$ and $\frac{p}{p-\alpha_2}$ in equation (59a) yields:

$$i = \frac{E}{2bL}(\epsilon^{\alpha_1 t} - \epsilon^{\alpha_2 t})$$

or

$$i = \frac{E}{bL}\epsilon^{-at}\left(\frac{\epsilon^{bt} - \epsilon^{-bt}}{2}\right) \tag{60}$$

Equation (60) reduces to one of the following forms

$$i = \frac{E}{bL}\,\epsilon^{-at}\sinh bt$$

or

$$i = \frac{E}{bL}\,\epsilon^{-at}\sin \beta t$$

depending upon whether

$$\frac{R^2}{4L^2} > \text{or} < \frac{1}{LC}$$

The critical case, that is, when $\frac{R^2}{4L^2} = \frac{1}{LC}$, requires special consideration as it did before.

Condenser Charge. *Case I:*

$$q = \int_0^t i\,dt = \int_0^t \frac{E}{bL}\,\epsilon^{-at}\sinh bt\,dt$$

$$q = EC\left[1 - \epsilon^{-at}\left(\frac{a}{b}\sinh bt + \cosh bt\right)\right] \tag{61}$$

An inspection of equation (61) shows that the condenser charge increases

from zero to a final value EC. The nature of the variation is similar to that shown in Fig. 11.

Case II:

$$q = \int_0^t idt = \frac{E}{\beta L} \int_0^t \epsilon^{-at} \sin \beta t dt$$

$$q = EC\left[1 - \frac{\epsilon^{-at} \sin (\beta t + \sigma)}{\beta \sqrt{LC}}\right] \tag{62}$$

where $\sigma = \tan^{-1} \frac{\beta}{a}$.

Interpretation of equation (62) reveals the nature of the variation. When $t = 0$ the condenser charge is zero. For values of $(\beta t + \sigma)$ greater than π and less than 2π, q becomes greater than EC. It reaches its steady value, EC, after going through a damped variation about that value. q is equal to EC at those times when $\beta t = (n\pi - \sigma)$, where n is any integer. The general nature of the variation is shown in Fig. 12.

Counter-Voltages. *Case I:* The expressions for the three counter-voltages are:

$$Ri = \frac{ER}{bL} \epsilon^{-at} \sinh bt \tag{63}$$

$$L\frac{di}{dt} = E\epsilon^{-at}\left(\cosh bt - \frac{a}{b} \sinh bt\right) \tag{64}$$

$$\frac{q}{C} = E\left[1 - \epsilon^{-at} (\cosh bt + \frac{a}{b} \sinh bt)\right] \tag{65}$$

The sum of the right-hand members of equations (63), (64), and (65) equals E. This serves as a check upon the various mathematical manipulations that have been employed in arriving at the expressions for i and q. The modes of variation of the component voltages are shown in Fig. 14.

Case II: The expressions for the component counter-voltages are:

$$Ri = \frac{ER}{\beta L} \epsilon^{-at} \sin \beta t \tag{66}$$

$$L\frac{di}{dt} = -\frac{E\epsilon^{-at}}{\beta\sqrt{LC}} \sin (\beta t - \sigma) \tag{67}$$

$$\frac{q}{C} = E\left[1 - \frac{\epsilon^{-at} \sin (\beta t + \sigma)}{\beta\sqrt{LC}}\right] \tag{68}$$

The sum of the three equations (66), (67), and (68) should again equal E and serve as a check upon the mathematical developments. Fig. 15

shows the general nature of the variations of these voltages with respect to time.

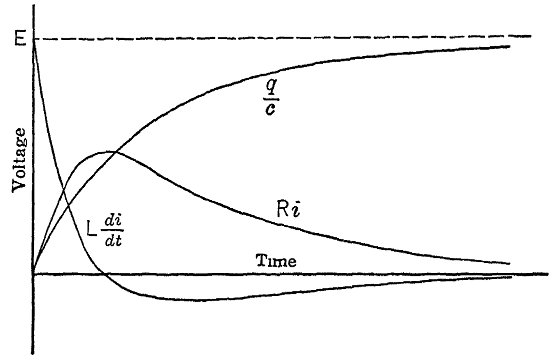


FIG. 14.—Modes of variation of the component counter-voltages in an *RLC* circuit when $\frac{R^2}{4L^2} > \frac{1}{LC}$.

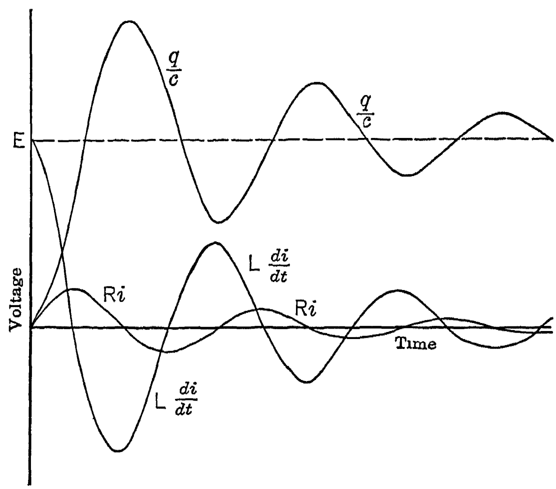


FIG. 15.—Modes of variation of the component counter-voltages in the *RLC* circuit when $\frac{R^2}{4L^2} < \frac{1}{LC}$.

Case III: A detailed analysis of the variations when $\frac{R^2}{4L^2} = \frac{1}{LC}$ is reserved for one of the exercises at the close of the chapter.

Energy and Power. *Case I:* During the initial period, which will be considered as the time from $t = 0$ to $t = \frac{1}{b}\tanh^{-1}\left(\frac{b}{a}\right)$, the energy

delivered to the circuit divides into three distinct components. Part of the electrical energy is transformed into heat, part of it is stored in the magnetic field, and the remainder is stored in the electrostatic field. After the current has reached its maximum value the energy that was stored in the magnetic field begins to return to the circuit. Energy continues to be dissipated in the form of heat and likewise continues to be stored in the electric field until the current decreases to zero.

The power consumed by the resistance is: Ri^2

The power delivered to the magnetic field is: $L\dfrac{di}{dt}i$

The power delivered to the electrostatic field is: $\dfrac{q}{C}i$

In expanded form the above expressions are somewhat cumbersome, but the general nature of the variations may be noted by examining the products of the individual counter-voltages and the current.

The total power delivered to the circuit is, of course:

$$Ei = \frac{E^2}{BL}\epsilon^{-at}\sinh bt \tag{69}$$

Case II: The expressions for power taken by the circuit parts in Case II may be obtained as outlined above for Case I. It should be noted that energy is, at times, supplied by the energy-storing devices of the circuit and that interchanges of energy between the electromagnetic and electrostatic fields occur periodically. The rate at which energy is supplied to the entire circuit, namely, the power delivered from the source, is the product of current and impressed voltage:

$$Ei = \frac{E^2}{\beta L}\epsilon^{-at}\sin\beta t \tag{70}$$

A simple analysis can be made of the commercial significance of transients in *RLC* circuits by assuming the resistance so small as to be negligible. In this case the circuit oscillates without any loss due to current flow, and the energy stored in the inductance when the current is a maximum, $\frac{1}{2}LI_m^2$, is equal to the energy stored in the condenser $\frac{1}{2}CE_m^2$ when the voltage across it is a maximum. The equality exists because it is the same stored energy which alternately appears in the inductance and in the condenser. Therefore

$$LI_m^2 = CE_m^2$$

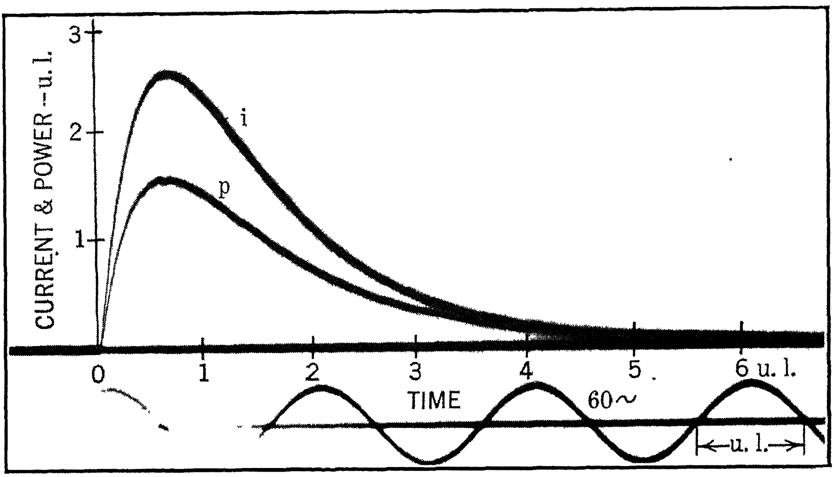
and

$$\frac{E_m^2}{I_m^2} = \frac{L}{C} \quad \text{or} \quad \frac{E_m}{I_m} = \sqrt{\frac{L}{C}}$$

Since this relationship between maximum transient voltage and maximum transient current has the nature of an impedance, it is known as "natural impedance" or "surge impedance," and its reciprocal $\sqrt{C/L}$ is called natural or surge admittance. With this constant known, either current or voltage can be calculated if the other is known.

Oscillographic Verification. *Case I:* With $R = 27.5$ ohms, $L = 0.07$ henry, and $C = 450$ microfarads, the RLC circuit is non-oscillatory.

$$\frac{R^2}{4L^2} > \frac{1}{LC}$$



OSCILLOGRAM 5.

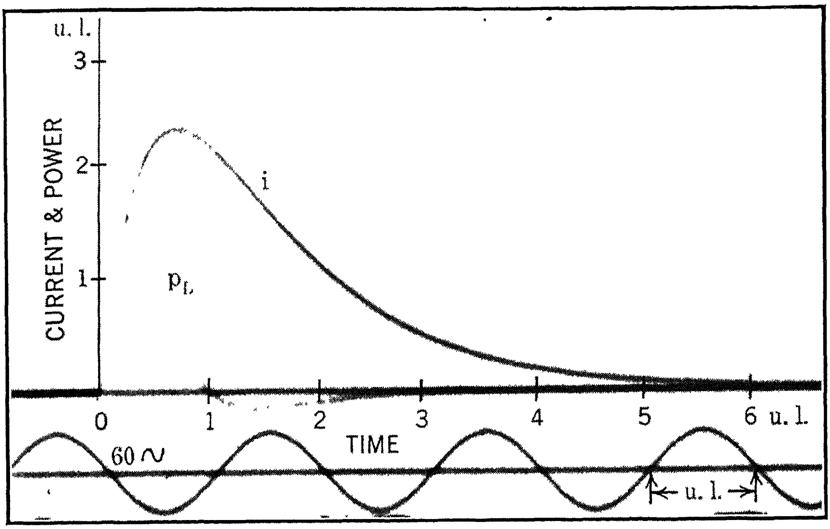
Current and total power in the RLC circuit. Case I.
$E = 85$ volts. $R = 27.5$ ohms. $L = 0.07$ henry. $C = 450\mu$f.
i calibration $= 0.91$ amp. per u. l.
p calibration $= 127$ watts per u. l.

The current and power variations in a circuit having such parameters are shown in Oscillogram 5. Both graphs follow the general $K\epsilon^{-at}\sinh bt$ form. A comparison of the i variation with the plotted solution shown in Fig. 11 will reveal the nicety with which the experimentally determined values agree with the theoretical. The time at which the current and power reach their maximum values is, theoretically, 0.0052 second. This corresponds with the value shown on Oscillogram 5.

With constant applied voltage the total power is directly proportional to the current. If it were possible to separate the self-inductance of the circuit completely from the resistance it would be possible to show

the rate at which energy is stored in the magnetic field and the rate at which that energy is returned to the electric circuit. Oscillogram 6 is an attempt to show $L\frac{di}{dt}\times i$. Actually, of course, the power variation in the inductance coil embraces the following:

$$L\frac{di}{dt}\times i + i^2 r_{\text{coil}}$$



OSCILLOGRAM 6.

Current and inductive power in the *RLC* circuit. Case I.

E = 85 volts. R = 27.5 ohms. L = 0.07 henry. C = 450μf.

i calibration = 0.91 amp. per u. l.

p_L calibration = 43 watts per u. l.

The other component powers, namely $(R - r_{\text{coil}})i^2$ and $(q/C)\times i$ may also be shown in a qualitative manner by means of the watt-galvanometer. However, the component relationships involved in the *RLC* circuit may be shown more accurately by oscillograms of the three component voltages, owing to the fact that a potential-galvanometer requires much less current to operate than a watt-galvanometer.

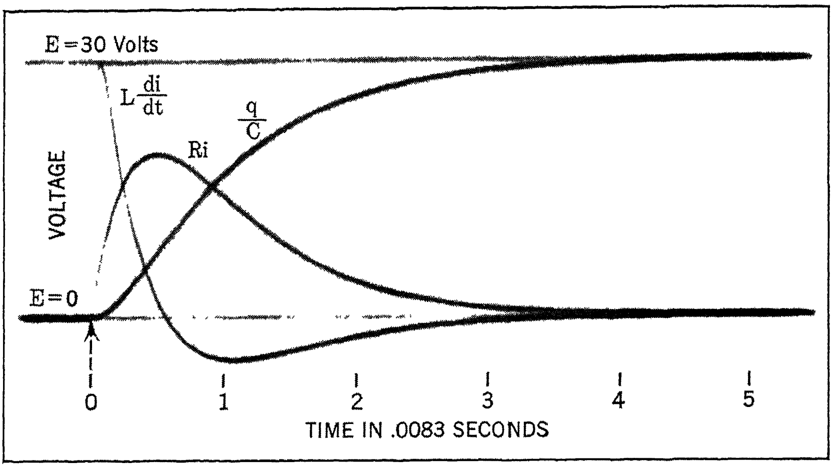
Oscillogram 7 illustrates the nature of the three component voltages which are operating to balance the applied emf. The Ri component is proportional to current (not shown), and the $L\frac{di}{dt}$ component is propor-

tional to the rate of change of the current. The q/C component follows the general trend of the theoretical q curve.

Case II: Oscillograms 8 and 9 illustrate the nature of the current and total power variations when:

$$\frac{R^2}{4L^2} < \frac{1}{LC}$$

The manner in which an increase in the resistance of the circuit affects



Oscillogram 7.

Component voltages in the RLC circuit. $\frac{R^2}{4L^2} > \frac{1}{LC}$.

E = 30 volts. R = 33.5 ohms. L = 0.056 henry. C = 317μf.

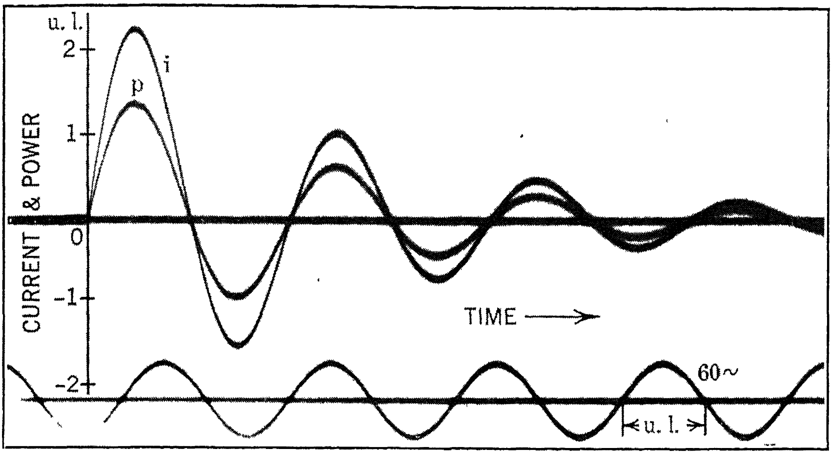
Ri, $L\frac{di}{dt}$, and $\frac{q}{C}$ calibrations are approximately equal.

the decrement is illustrated. In this particular case the numerical decrement changes from 0.54 in Oscillogram 8 to 0.75 in Oscillogram 9. However, the change in frequency is scarcely discernible on the oscillograms.

$$f_{\text{Osc } 8} = \frac{1}{6.28}\sqrt{99{,}500 - 1540} = 49.8 \text{ cycles per second}$$

$$f_{\text{Osc. } 9} = \frac{1}{6.28}\sqrt{99{,}500 - 4510} = 49.0 \text{ cycles per second}$$

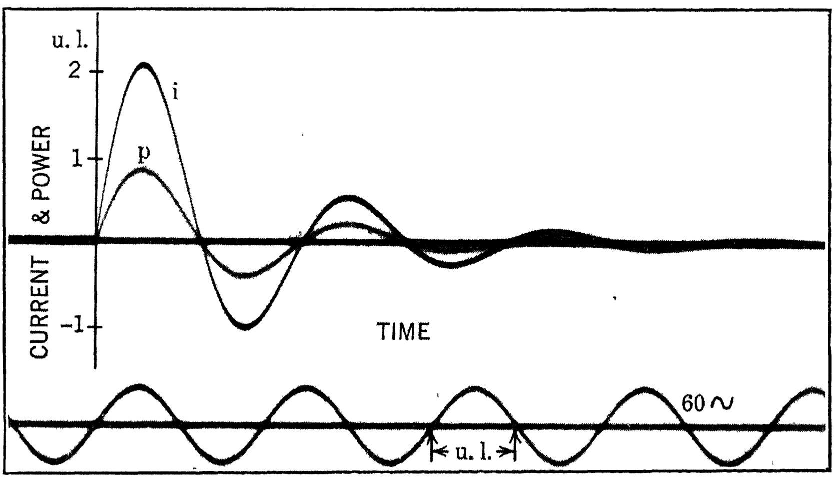
The first maximum current occurs at t = 0.0046 second in Oscillogram 8 as compared with 0.0044 second in Oscillogram 9.



OSCILLOGRAM 8.

Current and total power in the *RLC* circuit. $\frac{R^2}{4L^2} < \frac{1}{LC}$.

$E = 42$ volts. $R = 5.3$ ohms. $L = 0.067$ henry. $C = 150\mu$f.
i calibration $= 0.72$ amp. per u. l.
p calibration $= 50$ watts per u. l.



OSCILLOGRAM 9.

Current and total power in the *RLC* circuit. $\frac{R^2}{4L^2} < \frac{1}{LC}$.

Similar to Oscillogram 8 except for added resistance.
$E = 42$ volts. $R = 9$ ohms. $L = 0.067$ henry. $C = 150\mu$f.
i calibration $= 0.72$ amp. per u. l.
p calibration $= 74$ watts per u. l.

By placing potential galvanometers across the lumped resistance, across the inductance coil, and across the condenser, a qualitative indication of component voltages may be obtained. The circuit arrangement shown in Fig. 16 indicates the manner in which the original RLC circuit is disturbed by the shunting potential-galvanometers. However, the general trends of the component voltage variations are well illustrated by Oscillogram 10.

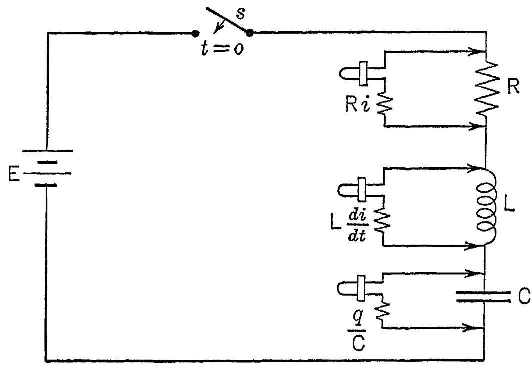
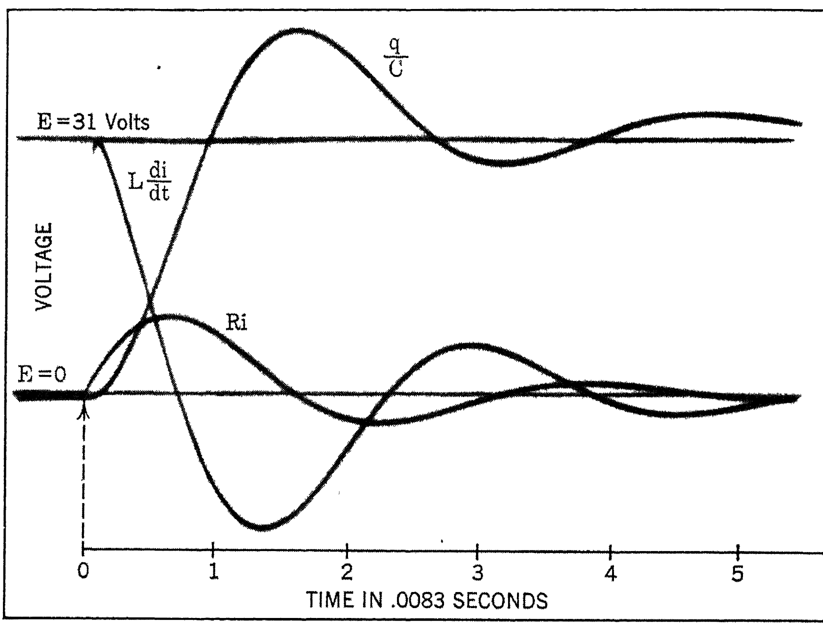


FIG. 16.—Circuit arrangement for obtaining component voltages.



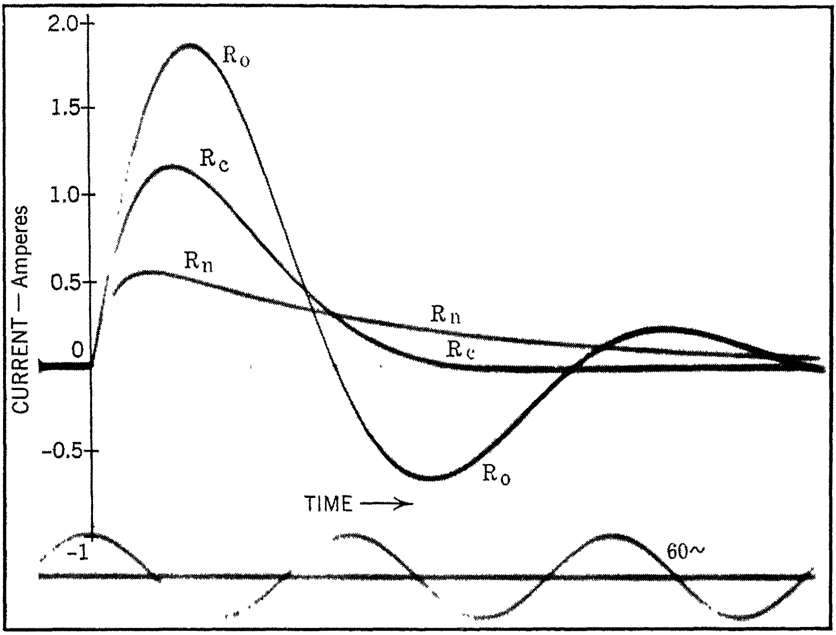
OSCILLOGRAM 10.

Component voltages in the RLC circuit. $\frac{R^2}{4L^2} < \frac{1}{LC}$.

$E = 31$ volts. $R = 7.5$ ohms. $L = 0.056$ henry. $C = 317\mu$f.

Ri, $L\frac{di}{dt}$, and $\frac{q}{C}$ calibrations are approximately equal.

The current variations in the *RLC* circuit, with different values of R, are superimposed in Oscillogram 11. The self-inductance and capacitance are the same for each graph. The R_n graph is for the case of $R_n > \sqrt{4L/C}$; the R_0 graph is for the case of $R_0 < \sqrt{4L/C}$; and the



OSCILLOGRAM 11.
Current variations in the *RLC* circuit for different values of R,—L and C remaining the same.
$E = 30\ 5$ volts. $L = 0.056$ henry. $C = 390\mu$f.
$R_n = 47.5$ ohms. $R_c = 19$ ohms. $R_0 = 7$ ohms.

R_c graph is for the case of R_c approximately equal to $\sqrt{4L/C}$. Numerically:

$$R_n = 47.5 \text{ ohms}$$

$$R_0 = 7.0 \text{ ohms}$$

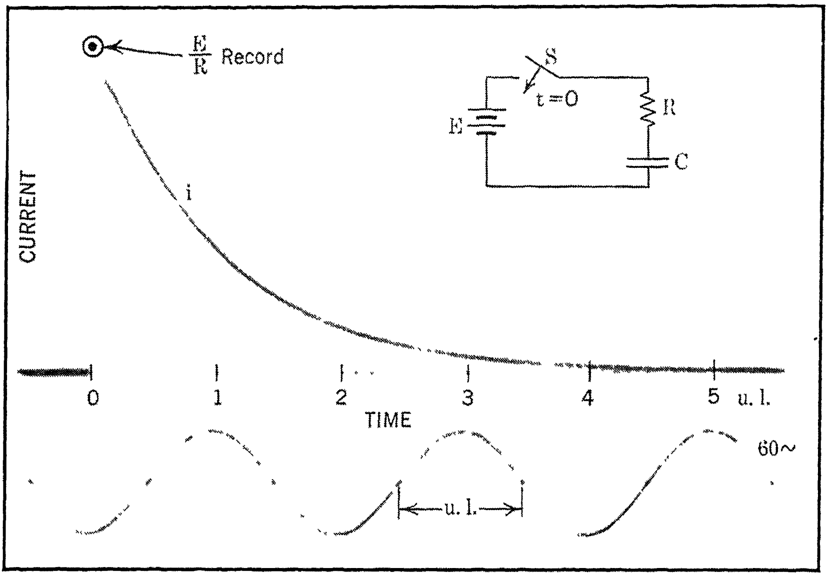
$$R_c = 19.0 \text{ ohms}$$

The actual critical resistance for $L = 0.056$ henry and $C = 390$ microfarads is 24 ohms. Therefore the graph marked R_c is, theoretically, within the limits of the oscillatory case. Its oscillation is barely discernible and serves to illustrate the smoothness with which a change

from a non-oscillatory condition to an oscillatory condition actually occurs.

EXERCISES

1. What is the time constant of the circuit shown in connection with Oscillogram 1? Does the value of i at t_c correspond to 63.2 per cent of the maximum value of i shown on the oscillogram?



OSCILLOGRAM 12.

Oscillogram to be used in connection with Exercises 4 and 5.

i calibration = 0.029 amp. per u. l.

2. (*a*) Refer to Oscillogram 2. How many joules of energy are stored in the magnetic field at $t = 0.01$ second? Compare the value found with the final amount of energy that is stored in the magnetic field.

(*b*) At what time is the rate of energy delivered to the magnetic field a maximum? (Mathematical determination is expected.) Check your result against the value shown on Oscillogram 2.

3. The field circuit of a synchronous motor has a self-inductance of 40 henrys (assumed constant) and a resistance of 100 ohms. The exciting voltage is 120 volts. Find the length of time required for the field current to build up to 99 per cent of its E/R value, counting time from the instant of applying voltage to the field and neglecting the mutual inductive effects between the field winding and other windings on the machine.

4. Assume that the circuit shown in connection with Oscillogram 12 has zero self-inductance. $E = 8$ volts, $R = 105$ ohms, and $Q_0 = 0$. What is the value of C in microfarads? What is the time constant of the circuit?

5. The actual E/R value of the current is shown on Oscillogram 12 by the spot directly above $t = 0$. Give two reasons for the oscillographic discrepancy near $t = 0$.

6. A potential difference of 100 volts is impressed across an RC branch at $t = 0$. $R = 100$ ohms, $C = 200$ microfarads, and $Q_0 = + 0.04$ coulomb. Graph i and q with respect to time. NOTE: The algebraic sign of Q_0 indicates the relative polarities of the impressed voltage, E, and the initial condenser voltage. See page 24.

7. Given: the applied emf and the circuit parameters as indicated below Oscillogram 11. Calculate the three maximum values of current from theoretical considerations. With the aid of the current scale determine the maximum values of current directly from the oscillogram, and compare the two sets of results.

8. Show that the numerical decrement is equal to $\left(1 - \epsilon^{-\frac{R}{2Lf}}\right)$. See equation (53).

9. Show that the logarithmic decrement is $\dfrac{R}{2Lf}$.

10. Given: the expression for current,

$$i = \frac{E}{\beta L} \epsilon^{-\alpha t} \sin \beta t$$

Evaluate the right-hand member as β approaches zero.

11. Prove the validity of equation (61) and of equation (62) by actual integration and substitution of limits.

12. Add the right-hand members of equations (63), (64), and (65). Explain the physical significance of the result.

13. Prove the validity of equations (67) and (68).

14. Show that the sum of the right-hand members of equations (66), (67), and (68) is equal to E.

15. A potential difference of 42 volts is to be applied to an RLC branch at $t = 0$. $R = 5.3$ ohms, $L = 0.067$ henry, $C = 150$ microfarads.

(a) Calculate the time at which the first positive maximum current occurs. At what fractional part of the cycle does the positive maximum current occur?

(b) Calculate the magnitude of the above current.

(c) Calculate the magnitudes of the two succeeding positive maxima of current.

(d) What is the value of the numerical decrement?

(e) What is the value of the logarithmic decrement?

(f) What is the frequency of oscillation? What is the length of time of one cycle?

(g) Check the above calculated values against those shown by Oscillogram 8.

(h) Graph the current and the condenser voltage with respect to time.

16. A potential difference of 85 volts is to be impressed across an RLC branch at $t = 0$. $L = 0.07$ henry, $C = 450$ microfarads, $R = \sqrt{4L/C}$.

(a) Calculate the time at which the maximum current occurs.

(b) What is the magnitude of the maximum current in amperes?

(c) What length of time, approximately, is required for the current to recede to one per cent of its maximum value?

(d) Plot the current variation with respect to time.

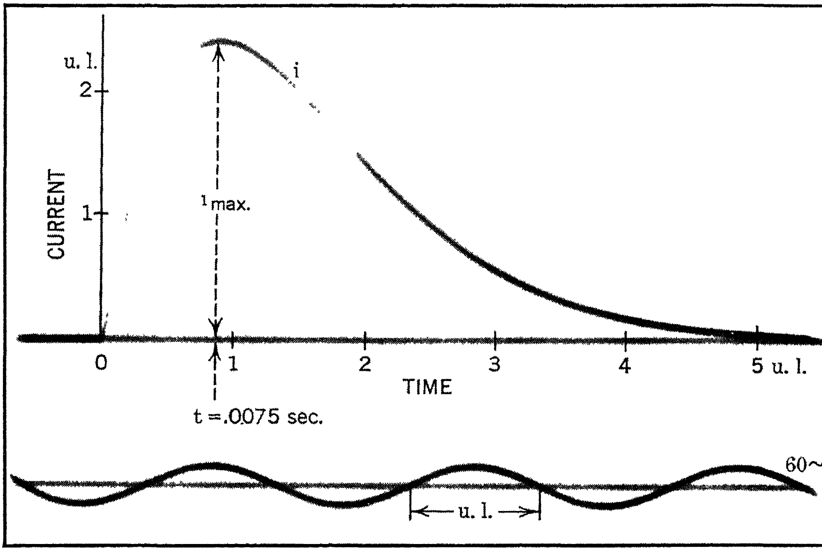
(e) Show the Ri, the $L\dfrac{di}{dt}$, and $\dfrac{q}{C}$ components on the same plot.

(*f*) What is the maximum number of joules stored in the magnetic field at any one time?

(*g*) What amount of energy is transformed into heat during the transient period?

(*h*) What amount of energy is finally stored in the electrostatic field of the condenser?

(*i*) What is the net amount of energy delivered to the *RLC* branch from the source over the entire transient period?



OSCILLOGRAM 13.

Current variation in a particular *RLC* branch when a constant potential difference is applied at $t = 0$. $Q_0 = 0$. Oscillogram to be used in connection with Problem 17.

17. Given the response of a particular *RLC* branch as shown in Oscillogram 13. Determine the numerical values of one set of circuit parameters that will give a similar response. If it is desired to reproduce the particular response shown in Oscillogram 13 using a fixed C of 100 microfarads, what values of R and L must be employed? Hint—Try Case III relations.

18. Prove that if a condenser is charged and then discharged for a useful purpose the efficiency cannot be greater than 50 per cent irrespective of the capacitance, the inductance, the resistance, or the voltage which is employed.

CHAPTER III

PARTICULAR BOUNDARY CONDITIONS

Classification.—The transient disturbances due to the application of a potential difference to circuits initially at rest have been considered. Many important transient manifestations occur, however, under boundary conditions other than initial rest. For purposes of classification in the present chapter these are divided into three groups:

A. Subsidence Transients.
B. Transition Transients.
C. Compound Transients.

The groups are not absolutely distinct in that certain particular cases will be found that overlap two of the groups.

Subsidence transients include the variations in current, voltage, and power as the circuit comes to rest from an initially active state. Examples of the boundary conditions are: $i = I_0$ at $t = 0$, and $q = Q_0$ at $t = 0$. In the latter case the circuit is considered to be potentially active at $t = 0$ by virtue of the initial condenser charge. In the present classification, only those cases where I_0 is a steady-state current and Q_0 is a fixed initial condenser charge are considered as subsidence transients. Changes from one set of steady-state values to a second set of steady-state values due to a sudden change in the circuit parameters are classified, for the want of a better name, as transition transients. Compound transients include those cases where the circuit, while still in a period of transition from one disturbance, is subjected to a second and possibly a third disturbance.

A. SUBSIDENCE TRANSIENTS

THE *RL* CIRCUIT

It is assumed that a steady current is passing through the circuit shown in Fig. 1 at $t = 0$ and that the switch s is changed from position a to position b at that time.

Physical Consideration.—When the current decays or decreases in value the surrounding magnetic field also decreases in intensity. The

decrease in the intensity of the magnetic field induces a component voltage in the circuit, and in accordance with Lenz's law the direction of the induced voltage is such as to oppose the change in flux linkage that is actually occurring. Therefore the induced voltage acts in the same direction through the RL branch as did the original driving voltage.

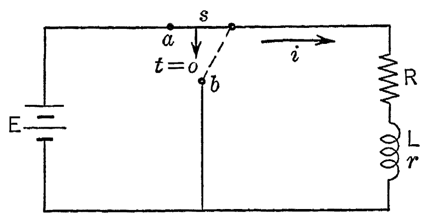


FIG. 1.—The RL discharge circuit.

Mathematical Development.—With the switch in position b the expression for dynamic equilibrium is:

$$Ri + L\frac{di}{dt} = 0 \tag{1}$$

from which:

$$i = A\epsilon^{-\frac{Rt}{L}} \tag{2}$$

Since:

$$i = \frac{E}{R} \quad \text{at } t = 0$$

$$A = \frac{E}{R} \tag{3}$$

The expression for current through the RL branch is, therefore:

$$i = \frac{E}{R}\epsilon^{-\frac{Rt}{L}} \tag{4}$$

From this equation it is evident that the current decays exponentially. The curve approaches the time axis asymptotically, that is, it becomes smaller and smaller but theoretically is not zero until an infinite time has elapsed. A photographic record of the decay current in the RL circuit is shown in Oscillogram 1.

Operational Solution.—The operational form for equation (1) is:

$$i(R + Lp) = 0$$

from which:

$$i = \frac{0}{R + Lp}$$

As this form of expression does not lend itself directly to an operational solution, an indirect method of solution must be applied. The general scheme is to determine

the current that will flow as a result of the application of $-E$ voltage to the circuit under the assumption of initial rest, and then add this current to the steady-state current resulting from the $+E$ voltage. By the principle of superposition the actual current in the circuit is the sum of the original current, $I_0 = +E/R$, and the current that results from the application of the $-E$ voltage. The insertion of the $-E$ voltage in the circuit shown in Fig. 2 is equivalent to the short-circuiting arrangement shown in Fig. 1.

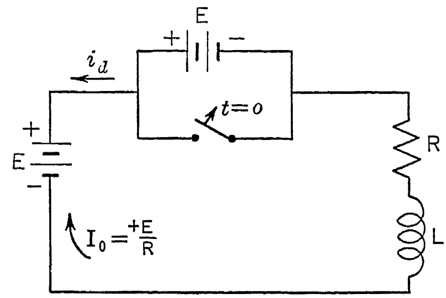


FIG. 2.—Schematic method of neutralizing the driving voltage thereby simulating a typical discharge circuit.

Under these conditions the actual current can be expressed by:

$$i = I_0 + i_d$$

where i is the actual decay current through R and L.

I_0 is the steady-state current due to $+E$ and is equal to E/R.

i_d is the current " due to " the application of the $-E$ voltage.

In Chapter II it was shown that:

$$i_d = -\frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right)$$

Therefore adding I_0 and i_d gives:

$$i = +\frac{E}{R} - \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right)$$

$$= \frac{E}{R}\epsilon^{-\frac{Rt}{L}}$$

Time Constant.—The rapidity of current decay is fixed by the value of the exponent. Circuits may be compared by noting the amount of time required for the current to decrease to one ϵth of its initial value. The time which must elapse to accomplish this is known as the time constant and is equal to:

$$t_c = \frac{L}{R}$$

as in the case of the building-up process. When $t = L/R$ the exponential factor becomes equal to:

$$\epsilon^{-\frac{R}{L}\cdot\frac{L}{R}} = \epsilon^{-1} = 0.368$$

Either a high value of resistance or low value of self-inductance (other things being equal) will make the decay of current rapid.

Power and Energy Considerations.—As the current decreases from its initial E/R value, energy is being transformed into heat at a rate which is equal to:

$$p_R = Ri^2 = \frac{E^2}{R}\epsilon^{-\frac{2Rt}{L}} \tag{5}$$

The total amount of energy transformed into heat during the subsidence period is:

$$W_R = \int_0^\infty \frac{E^2}{R}\epsilon^{-\frac{2Rt}{L}}\,dt \tag{6}$$

$$= \frac{E^2L}{2R^2} = \frac{I_0{}^2L}{2} \tag{7}$$

The above is, of course, the energy that was originally stored in the magnetic field. The rate at which energy is returned to the electric circuit from the magnetic field is:

$$p_L = L\frac{di}{dt}i = -L\cdot\frac{E}{R}\cdot\frac{R}{L}\epsilon^{-\frac{Rt}{L}}\cdot\frac{E}{R}\epsilon^{-\frac{Rt}{L}}$$

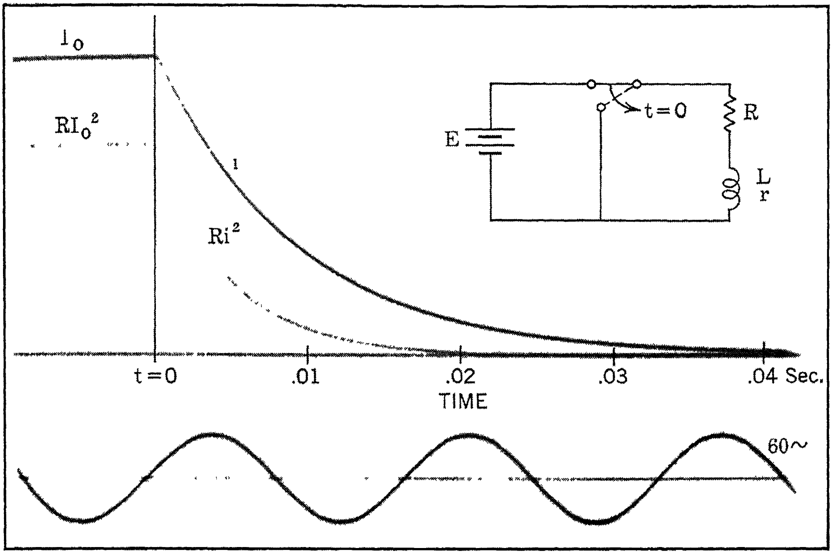
$$= \frac{-E^2}{R}\cdot\epsilon^{-\frac{2Rt}{L}} \tag{8}$$

With respect to energy delivered to the magnetic field, p_L is a minus quantity.

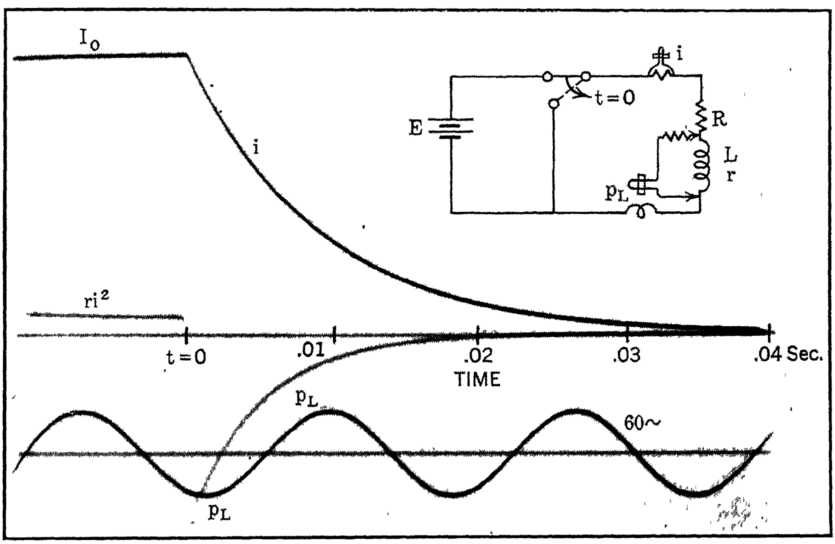
A photographic record of p_R is shown on Oscillogram 1. The relative rates of subsidence of i and p_R are clearly shown.

Oscillogram 2 illustrates the manner in which the decaying magnetic field returns energy to the electrical circuit. Prior to $t = 0$ the p galvanometer records the steady state $I_0{}^2r$, where r is the resistance of the inductance coil. After $t = 0$ it records $\left(L\frac{di}{dt} + ri\right)i$. Neglecting the ri^2 loss in the L coil, the area under the p_L graph represents the energy returned to the circuit by the magnetic field.

Theory of Discharge Resistances.—An attempt to open suddenly a highly inductive circuit may result in a dangerously high induced voltage. Destructive arcs will appear at the blades of the opening switch unless some device is employed to prevent such formation. In order to overcome these difficulties, discharge resistances are often used where highly inductive circuits must be opened. Prior to opening the circuit, a resistance of suitable magnitude is placed in parallel with



OSCILLOGRAM 1.
Current and Ri^2 subsidence in a circuit having $(R + r)$ ohms resistance and L henrys inductance.
$E = 30$ volts. $R = 6.0$ ohms. $r = 0.5$ ohm. $L = 0.057$ henry.



OSCILLOGRAM 2.
Current and inductive power during the period of discharge. The p_L graph includes the i^2r loss of the inductance coil. Negative values of p_L indicate power returned to the circuit.
$E = 30$ volts. $R = 6.0$ ohms. $r = 0.5$ ohm. $L = 0.057$ henry.

the inductive branch. Fig. 3 shows the arrangement. During the process of opening switch s, but before the actual opening occurs, s_1 is closed, thereby shunting the RL branch with R_d. The ER_d loop possesses a relatively small self-inductance, and in general no difficulty is experienced in breaking I_d. When switch, s, is opened, I_0 establishes itself in the R_d branch almost instantly. I_0 is the current in the highly inductive branch at $t = 0$. Having established itself through the R_d branch, I_0 decays at a rate which is governed by the ratio, $\frac{R + R_d}{L}$. The energy stored in the magnetic field of the RL branch may now dissipate itself in the heat loss in the RLR_d loop rather than in the form of destructive arcs at the blades of the opening switch.

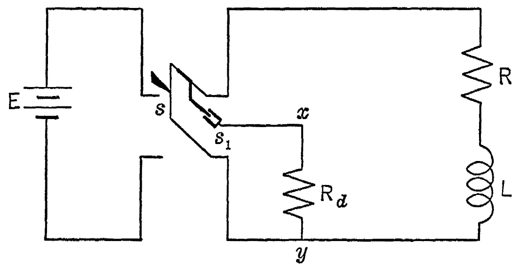


FIG. 3.—An ERL circuit equipped with discharge resistance.

Assuming that I_d can be interrupted in an infinitely short period of time, the expression for dynamic equilibrium after the opening of switch s is:

$$(R + R_d)i + L\frac{di}{dt} = 0$$

from which:

$$i = A\epsilon^{-\frac{(R+R_d)t}{L}} \tag{9}$$

Since:

$$i = I_0 = \frac{E}{R} \qquad \text{at } t = 0$$

$$A = \frac{E}{R}$$

and

$$i = \frac{E}{R}\epsilon^{-\frac{(R+R_d)t}{L}} \tag{10}$$

The voltage that appears across the terminals of the RL branch is iR_d. iR_d is of polarity opposite to that of the original driving voltage E. Equation (10) is the expression for current through the RLR_d loop in the direction of positive I_0 through the RL branch. With respect to

the original polarity of current through the R_d branch, i is actually negative through that branch and the potential drop is from y to x (Fig. 3) after $t = 0$ if it was from x to y before $t = 0$. The external voltage drop that appears across the terminals of the inductive branch is of the same magnitude as that across the resistance R_d, namely:

$$\begin{aligned} V_{xy} &= -\ iR_d \\ &= -\ \frac{ER_d}{R}\epsilon^{-\frac{(R+R_d)t}{L}} \end{aligned} \tag{11}$$

V is of greatest magnitude at $t = 0$ and is equal to $-ER_d/R$. When $R_d = R$ the potential difference across the terminals of the RL branch at $t = 0$ is equal in magnitude to its value just prior to $t = 0$ and is of opposite polarity.

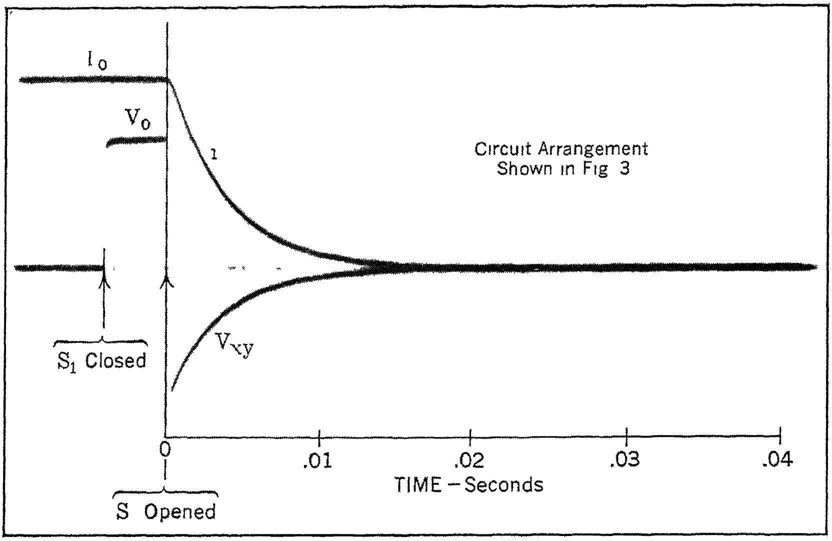
In order to lessen the magnitude of the original I_d, R_d may be increased to several times the value of R and, generally speaking, the RLR_d circuit is still well within the limits of safe operation. From equation (11) it is evident that R_d should not be made too large. Opening an inductive circuit without a discharge resistance is equivalent to inserting a very large resistance in series with the original R and L. The potential difference that appears at the blades of the opening switch then becomes sufficiently high to establish arcs through which the discharge current passes.

Oscillograms 3 and 4 illustrate the action of a discharge resistance switch in bringing an RL circuit to rest. In Oscillogram 3 the discharge resistance is approximately equal to the resistance of the RL branch. The case of R_d being several times as large as R is illustrated in Oscillogram 4. The oscillograms show the effect of the insertion of R_d as well as the phenomena that follow the opening of switch s.

The *RC* Circuit

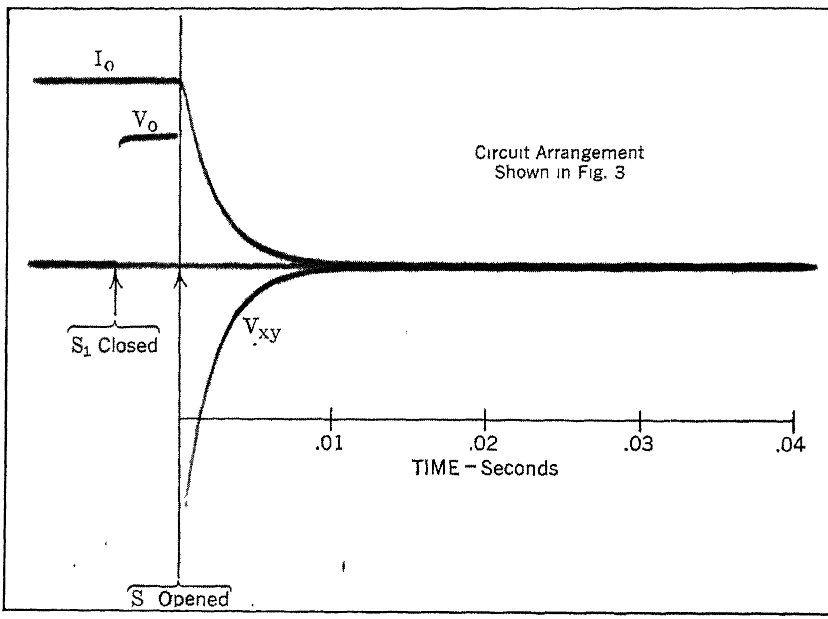
The circuit shown in Fig. 4 is assumed to be in static equilibrium at $t = 0$. A change of the switch from position a to position b results in the condenser discharging through the closed path.

Physical Considerations.—A brief analysis of Fig. 4 will suffice to show the nature of the condenser discharge current. Assume that the switch s has been closed in position a sufficiently long for the circuit to have reached static equilibrium. Since the condenser now exhibits a potential difference equal and opposite to the impressed voltage E, an initial current given by $I = E/R$ will flow through the short-circuited path established by moving the switch to b. As the current flows, how-



OSCILLOGRAM 3.

The i graph represents the subsidence of the current in the RLR_d loop of Fig. 3.
The V graph is the voltage drop across the discharge resistance R_d.
$E = 12.5$ volts. $R = 16$ ohms. $L = 0.056$ henry. $R_d = 16.5$ ohms.



OSCILLOGRAM 4.

Similar to Oscillogram 3 except for a larger value of discharge resistance.
Note the correspondingly larger voltage across R_d at $t = 0$.
$E = 12.5$ volts. $R = 16$ ohms. $L = 0.056$ henry. $R_d = 50$ ohms.

ever, the charge on the condenser is reduced, and with it the voltage. As the voltage is lowered the emf impressed upon the resistance becomes less and thus the current will be less than at first. The continuation of the flow of current further discharges the condenser and so still further reduces the voltage across the condenser. The rate of discharge therefore decreases, owing to the decreasing potential.

Mathematical Analysis.—With the switch in position b, it is evident that:

$$Ri + \frac{q}{C} = 0 \tag{12}$$

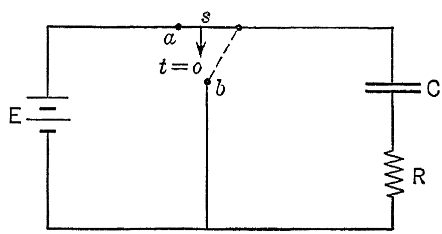


FIG. 4.—The RC discharge circuit.

The solution of equation (12) is similar to that given for equation (30) in Chapter II.

$$\log i = -\frac{t}{RC} + c_1 \tag{13}$$

$$i = A\epsilon^{-\frac{t}{RC}} \tag{14}$$

The initial boundary condition in the present case is that:

$$q = Q_0 = EC \qquad \text{at } t = 0$$

Upon substituting this boundary condition and the above expression for i in equation (12) it is found that:

$$A = -\frac{E}{R}$$

Therefore:

$$i = -\frac{E}{R}\epsilon^{-\frac{t}{RC}} \tag{15}$$

The charge on the plates of the condenser at any time during the discharge period is:

$$q = \int_0^t idt + Q_0$$

or

$$q = EC\epsilon^{-\frac{t}{RC}} - EC + Q_0$$

Since $Q_0 = EC$:

$$q = EC\epsilon^{-\frac{t}{RC}} \tag{16}$$

The circuit comes to rest when $\int idt$ is equal to $-Q_0$.

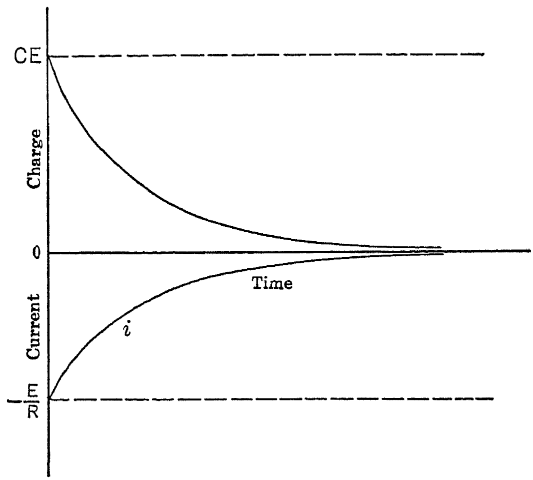


FIG. 5.—Charge and current variations in the RC discharge circuit.

Graphs of the i and q variations during the discharge period are shown in Fig. 5. It will be noted that the discharge current is similar in nature but opposite in sign to the charging current described in Chapter II. The graph of q during discharge is of course a subsidence curve as contrasted to the $K(1 - \epsilon^{-at})$ graph during the charging process.

Operational Solution.—An operational solution of the above problem may be effected very simply. The expression for dynamic equilibrium is:

$$Ri + \frac{\int_0^t idt}{C} + \frac{Q_0}{C} = 0$$

which in operational form is:

$$Ri + \frac{i}{pC} + \frac{Q_0}{C} = 0$$

or

$$i\left(R + \frac{1}{pC}\right) + \frac{Q_0}{C} = 0$$

$$i = -\frac{Q_0/C}{R + \frac{1}{pC}}$$

$$= -\frac{E}{R}\frac{1}{1 + \frac{1}{pRC}}\mathbf{1}$$

Substituting the exponential form for $\frac{1}{1 + \frac{a}{p}}\mathbf{1}$ gives:

$$i = -\frac{E}{R}\cdot\epsilon^{-\frac{t}{RC}}$$

It will be observed that the initial condenser voltage Q_0/C is written into the original voltage equation.

Time Constant.—The time constant, $t_c = RC$, as in the charging process is the time required for the current to drop to one ϵth of its initial value.

Power and Energy Considerations.—The energy initially stored in the electrostatic field is transformed into heat during the discharge period, and the rate at which the energy transformation occurs is:

$$p_R = Ri^2 = \frac{E^2}{R}\epsilon^{-\frac{2t}{RC}} \tag{17}$$

A summation of the above power over the complete period of discharge represents the total energy thus transformed.

$$W_R = \int_0^\infty \frac{E^2}{R}\epsilon^{-\frac{2t}{RC}}\,dt$$

$$= \frac{E^2C}{2} \tag{18}$$

At least two assumptions have been made throughout the present discussion that are physically unrealizable. The circuit has been

assumed to be void of self-inductance. The second assumption is that the condenser dielectric possesses no hysteresis. It is only under such ideal conditions that the energy stored in the electrostatic field is actually returned to the electrical circuit at the rate indicated by equation (17). Very close approaches to the ideal *RC* circuit can, however, be established in the laboratory.

The *RLC* Circuit

Mathematical Procedure.—It will be assumed that the circuit shown in Fig. 4 possesses an appreciable amount of self-inductance. The effect of short-circuiting the *RLC* branch by changing the switch to position *b* may be determined in a manner similar to that given above except that in the present case two sets of initial boundary conditions exist:

$$(1)\ q = Q_0 \quad \text{at } t = 0$$

$$(2)\ i = 0 \quad \text{at } t = 0$$

The expression for dynamic equilibrium must, of course, include the $L\frac{di}{dt}$ counter-voltage, thus:

$$L\frac{di}{dt} + Ri + \frac{q}{C} = 0 \tag{19}$$

Differentiating gives:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{20}$$

from which:

$$i = A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \qquad \alpha_1 \neq \alpha_2 \tag{21}$$

and:

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a + b$$

$$\alpha_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a - b$$

The details connected with the evaluation of α_1 and α_2 are described in Chapter II. The two constants of integration, A_1 and A_2, must be evaluated from the initial boundary conditions. If equation (19) is to have a unique solution of the form shown in (21), two independent relationships involving A_1 and A_2 must be found. It is quite immaterial what form these relationships take, provided they are independent and

one of them has been obtained by imposing the initial boundary conditions on the original equation or its equivalent. From equation (19):

$$\left(\frac{di}{dt}\right)_{t=0} = -\frac{Q_0}{LC}$$

Also:

$$\left(\frac{di}{dt}\right)_{t=0} = A_1\alpha_1 + A_2\alpha_2 = -\frac{Q_0}{LC} \tag{22}$$

And from equation (21):

$$A_1 + A_2 = 0 \tag{23}$$

Solving equations (22) and (23) simultaneously gives:

$$A_1 = -\frac{Q_0/C}{2bL}$$

$$A_2 = \frac{Q_0/C}{2bL}$$

The expression for the discharge current thus becomes:

$$i = -\frac{Q_0/C}{bL}\,\epsilon^{-at}\left(\frac{\epsilon^{bt} - \epsilon^{-bt}}{2}\right) \tag{24}$$

This expression is similar, except for the reversal of sign, to the expression for charging current and may take corresponding mathematical forms. Only the oscillatory case will be considered at this time; thus, if

$$Q_0 = EC \quad \text{and} \quad R^2 < \frac{4L}{C}$$

the expression for i becomes:

$$i = -\frac{E}{\beta L}\,\epsilon^{-at}\sin\beta t \tag{25}$$

where

$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

The negative sign of the current is with respect to the positive charging current. With respect to the discharge circuit only the opposite convention of signs may be employed. Oscillogram 5 is a photographic record of discharge current where the convention of signs is such that the initial discharge current is positive. The variation in the (R_1i^2) loss is also shown.

Condenser Discharge.—Employing the conventional negative sign for discharge current, the expression for the condenser charge at any instant is:

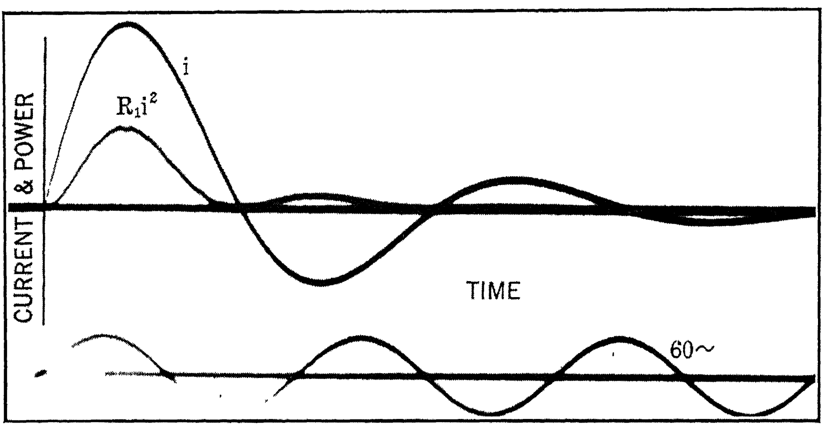
$$q = \int_0^t i dt + Q_0$$

$$q = -\frac{Q_0}{C\beta L}\int_0^t \epsilon^{-at} \sin \beta t dt + Q_0$$

$$q = + \frac{Q_0}{\beta\sqrt{LC}}\epsilon^{-at} \sin (\beta t + \sigma) \tag{26}$$

where

$$\sigma = \tan^{-1}\frac{\beta}{a}$$



Oscillogram 5.

Oscillatory discharge current in the *RLC* circuit together with power loss in the external resistance, R_1.

Q_0/C = 228 volts. R_1 = 10 ohms. r = 3.5 ohms. L = 0.093 henry. C = 160μf.

q has the value Q_0 at $t = 0$, and its variation thereafter is that of a damped sine wave about the zero axis. The circuit comes to rest when $\int_0^t i dt = -Q_0$. For a given value of L the rate at which the circuit comes to rest is governed by the amount of resistance in the discharge circuit.

Energy Interchanges During Discharge.—The source of energy is originally in the electrostatic field of the condenser. During the

oscillatory discharge, q periodically becomes zero, and at such times the energy stored in the condenser is equal to zero. From equation (26):

$$q = 0 \text{ at } t = \frac{n\pi - \sigma}{\beta}$$

where n is any positive integer.

When $t = \dfrac{n\pi - \sigma}{\beta}$, the discharge current, equation (25), is at maximum or minimum value, and the entire energy possessed by the circuit is stored in the magnetic field. The successive energy interchanges between the electrostatic field and the electromagnetic field decrease because of the transformation of energy into heat, namely, the $\int_0^t Ri^2dt$ loss. This particular phenomenon may be illustrated oscillographically

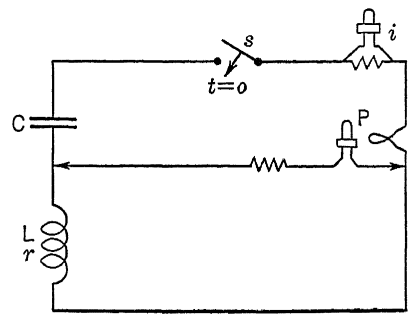
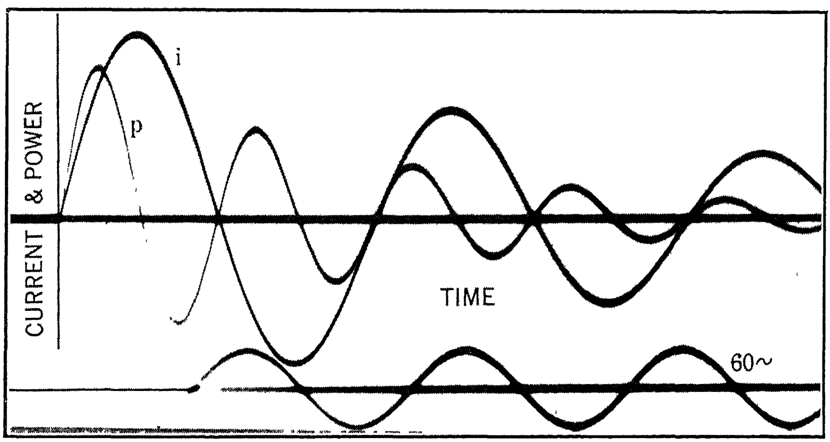


FIG. 6.—Method of photographing the power oscillations between the electrostatic and electromagnetic fields.



OSCILLOGRAM 6.

Oscillatory discharge current in the *RLC* circuit together with the power oscillations that take place between the electrostatic and electromagnetic fields. (See Fig. 6.) $Q_0/C = 228$ volts. $r = 4.0$ ohms. $L = 0.093$ henry. $C = 160\mu$f.

with the circuit arranged as in Fig. 6. The watt-galvanometer measures $\left(L\dfrac{di}{dt} + ir\right)i$, the rate at which energy is delivered to the Lr portion

of the discharge circuit. This is, of course, the rate at which the condenser supplies the energy. Oscilligram 6 illustrates the double frequency nature of the power variation. The initial discharge current is photographed positively. Neglecting the i^2rt loss the areas under the positive power loops represent energy delivered to the magnetic field, and the areas under the negative power loops represent energy returned to the condenser from the magnetic field.

B. TRANSITION TRANSIENTS

Example 1.—Short-circuiting R_2L_2 in an $ER_1L_1R_2L_2$ circuit.

The result of closing the switch s_1 in Fig. 7 is to short-circuit R_2 and L_2. Assuming that the current in the circuit, prior to the closing

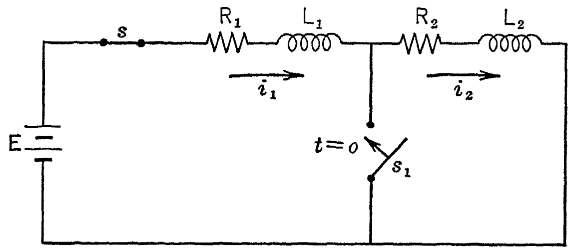


FIG. 7.—Short-circuiting R_2L_2 at $t = 0$.

of switch s_1, has reached its steady-state value, let it be required to find the expressions for i_1 and i_2.

The boundary condition is that:

$$i_1 = i_2 = \frac{E}{R_1 + R_2} \qquad \text{at } t = 0$$

After $t = 0$:

$$L_1 \frac{di_1}{dt} + R_1 i_1 = E \tag{27}$$

The solution of this type of equation has been given in Chapter I, and is:

$$i_1 = \frac{E}{R_1} + c_1 \epsilon^{-\frac{R_1 t}{L_1}} \tag{28}$$

Imposing the boundary condition upon equation (28)

$$c_1 = \frac{-E}{R_1}\left(\frac{R_2}{R_1 + R_2}\right) \tag{29}$$

$$i_1 = \frac{E}{R_1}\left[1 - \left(\frac{R_2}{R_1 + R_2}\right)\epsilon^{-\frac{R_1 t}{L_1}}\right] \tag{30}$$

Assuming that the resistance of the s_1 switch is equal to zero, the following relation holds after $t = 0$:

$$L_2 \frac{di_2}{dt} + R_2 i_2 = 0 \tag{31}$$

from which:

$$i_2 = c_2 \epsilon^{-\frac{R_2 t}{L_2}} \tag{32}$$

Since

$$i_2 = \frac{E}{R_1 + R_2} \qquad \text{at } t = 0$$

$$i_2 = \frac{E}{R_1 + R_2} \epsilon^{-\frac{R_2 t}{L_2}} \tag{33}$$

It will be observed that the current through the switch s_1 is the difference

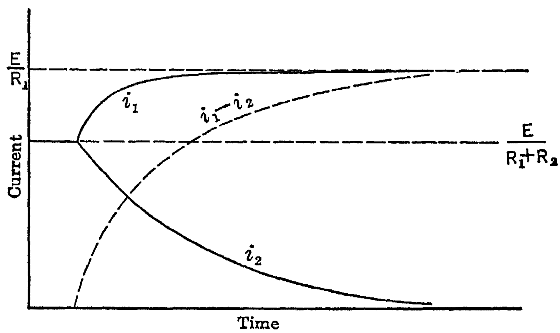


FIG. 8.—Plotted solutions of equations (30) and (33) for a particular set of circuit parameters.

between i_1 and i_2. The nature of the current transitions is shown in Fig. 8.

Example 2.—Inserting a condenser in an *ERL* circuit.

The effect of inserting a condenser into an original *RL* circuit may be determined by analyzing the arrangement shown in Fig. 9. With respect to the current, the transient is one of a subsidence nature; but with respect to energy storage, the transient might be classified under the present heading. It is assumed that the current through *RL* is at its E/R value at the time switch s_1 is opened, and time is reckoned from that instant.

The boundary conditions are:

$$(1)\quad i = \frac{E}{R} \qquad \text{at } t = 0$$

$$(2)\quad q = 0 \qquad \text{at } t = 0$$

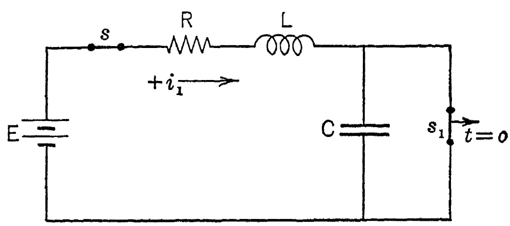


FIG. 9.—Inserting a condenser in an ERL circuit at $t = 0$.

For dynamic equilibrium:

$$Ri + L\frac{di}{dt} + \frac{q}{C} = E \tag{34}$$

or:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{35}$$

from which:

$$i = A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \qquad \alpha_1 \neq \alpha_2 \tag{36}$$

where

$$\alpha_1 = -a + b$$

$$\alpha_2 = -a - b$$

$$a = \frac{R}{2L}$$

$$b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

The boundary conditions imposed upon the original expression, equation (34), yield:

$$A_1\alpha_1 + A_2\alpha_2 = 0 \tag{37}$$

Since two constants of integration are involved another independent relationship between them must be established. Such may be had by imposing the first boundary condition upon equation (36).

$$A_1 + A_2 = E/R \tag{38}$$

Solving equations (37) and (38) simultaneously yields:

$$A_1 = \frac{E}{R}\frac{\alpha_2}{\alpha_2 - \alpha_1}$$

$$A_2 = -\frac{E}{R}\frac{\alpha_1}{\alpha_2 - \alpha_1}$$

$$i = \frac{E}{2Rb}\,\epsilon^{-at}\left\{(a + b)\epsilon^{bt} - (a - b)\epsilon^{-bt}\right\} \tag{39}$$

If $\dfrac{R^2}{4L^2} > \dfrac{1}{LC}$, equation (39) reduces to:

$$i = \frac{E}{Rb}\,\epsilon^{-at}\,(a \sinh bt + b \cosh bt) \tag{40}$$

In this case the current starts with its E/R value at $t = 0$ and decreases to zero without attaining negative values.

If $\dfrac{R^2}{4L^2} < \dfrac{1}{LC}$, equation (39) reduces to:

$$i = \frac{E}{R\beta}\,\epsilon^{-at}\,(a \sin \beta t + \beta \cos \beta t) \tag{41}$$

where

$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

In this case the current starts with its E/R value at $t = 0$ and reaches zero by way of a damped oscillation.

The energy originally stored in the magnetic field is $LI_0{}^2/2$. Inserting the condenser into the circuit ultimately reduces the electromagnetic energy to zero. At the end of the transient period, the $CE^2/2$ energy is stored in the electrostatic field, but in general, the final energy stored will not be equal to the energy originally stored in the electromagnetic field. Such, however, will be the case when $R = \sqrt{L/C}$.

C. COMPOUND TRANSIENTS

An analysis of a simple circuit will serve to illustrate the effect of a change in the circuit parameters while the current is yet in a period of transition from a previous disturbance. The circuit under consideration and the nature of the switching operations are shown in Fig. 10.

Mathematical Analysis.—Time will first be reckoned from the instant of closing switch s. As long as switch s_1 remains closed the expression for i is:

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right) \tag{42}$$

Equation (42) defines the transition of current from the time of closing switch s to the time of opening switch s_1. Let s_1 be opened at $t = t_1$. t_1 is now a new starting-point in the investigation.

$$\text{At } t_1 : i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt_1}{L}}\right) \tag{43}$$

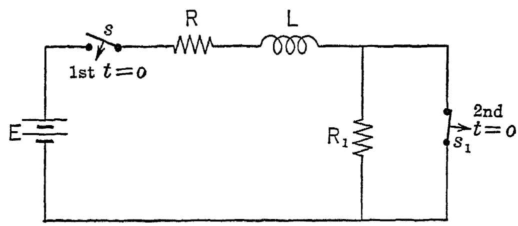


FIG. 10.—Inserting the resistance R_1 into an ERL circuit before the current has reached its steady-state value.

Equation (43) is, for the purposes of the second investigation, a boundary condition and a constant quantity. The expression for dynamic equilibrium after t_1 is:

$$(R + R_1)\, i + L\frac{di}{dt} = E \tag{44}$$

$$i = \frac{E}{R + R_1} + c_1 \epsilon^{-\frac{(R+R_1)t}{L}} \tag{45}$$

c_1 is the constant of integration and t is reckoned anew, that is, considered to be equal to zero at the time of opening switch s_1.

Imposing the boundary condition upon equation (45):

$$c_1 = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt_1}{L}}\right) - \frac{E}{R + R_1} \tag{46}$$

$$i = \frac{E}{R + R_1}\left(1 - \epsilon^{-\frac{(R+R_1)t}{L}}\right) + \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt_1}{L}}\right)\epsilon^{-\frac{(R+R_1)t}{L}} \tag{47}$$

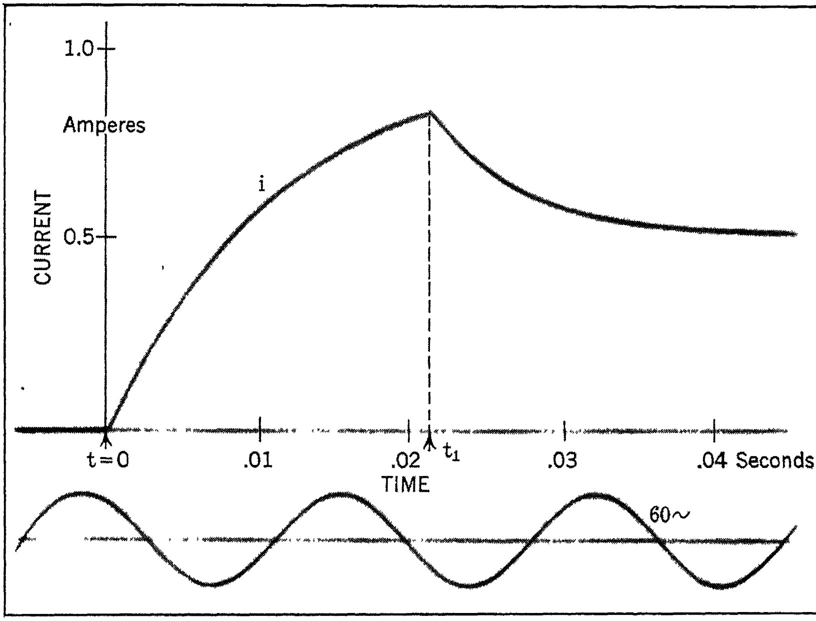
Equation (47) defines the transition of current at and after t_1. The current time graph is shown in Oscillogram 7 for the case of

$$\left(\frac{E}{R}\right)\left(1 - \epsilon^{-\frac{Rt_1}{L}}\right) > \frac{E}{R + R_1}$$

It is entirely possible, in this type of analysis, to count time from the first $t = 0$ throughout the entire discussion. In such case the expression for time after the opening of switch s_1 would be $(t - t_1)$.

Elimination of Resistance at $t = t_1$.—Oscillogram 8 illustrates the effect of eliminating a part of the circuit resistance while the current is in a period of transition from zero to its $\frac{E}{R_1 + R_2}$ value. The expression for current during the early period is

$$i_0 = \frac{E}{R_1 + R_2}\left(1 - \epsilon^{-\frac{R_1+R_2}{L}t}\right) \tag{48}$$

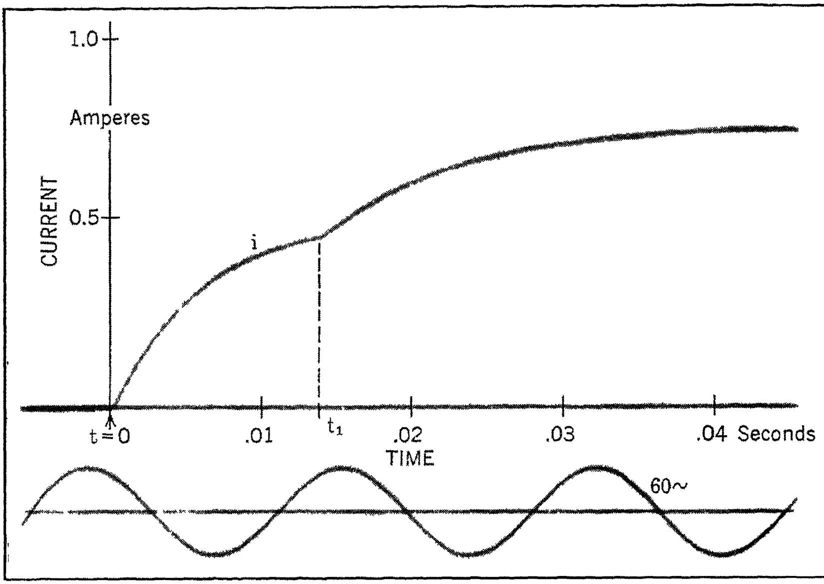


OSCILLOGRAM 7.

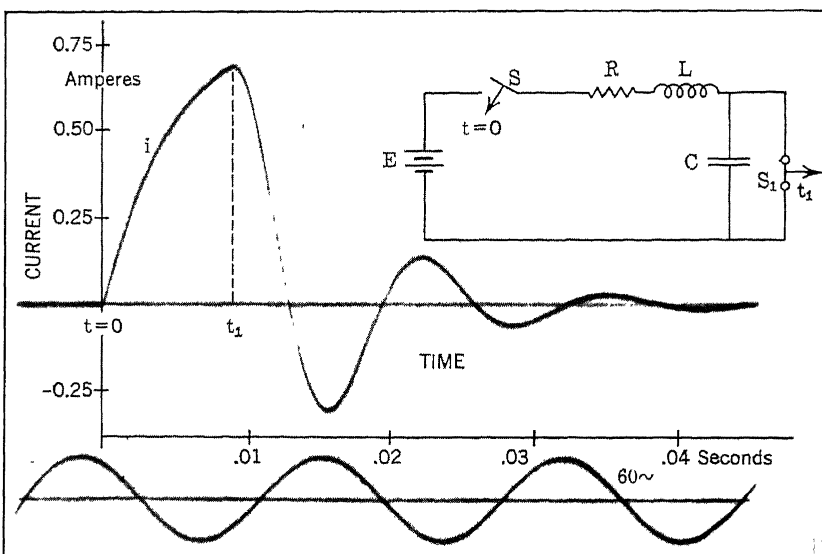
Compound transient caused by the switching operations indicated in Fig. 10.
$E = 10$ volts. $R_1 = 10$ ohms. $L_1 = 0.112$ henry. $R_2 = 9.25$ ohms.
R_2 inserted at t_1.

At time t_1, R_2 is eliminated by a short-circuiting switch. After t_1 the current rises to its E/R_1 value at a rate which is determined by the ratio of R_1 to L.

Condenser Inserted at $t = t_1$.—A compound transient effect which is caused by inserting a condenser into the series circuit at $t = t_1$ is illustrated by Oscillogram 9. During the early period the current rises toward its E/R value but is interrupted by the insertion of the condenser. The mathematical analysis of this case has been considered in detail in



OSCILLOGRAM 8.
Illustrating the effect of eliminating a part of the resistance of the $(R_1 + R_2)L$ circuit.
$E = 10$ volts. $R_1 = 7.2$ ohms. $L = 0.112$ henry. $R_2 = 12.8$ ohms.
R_1 eliminated at $t = t_1$.



OSCILLOGRAM 9.
Effect of inserting a condenser into an ERL circuit before i has reached its E/R value.
$E = 10$ volts. $R = 12.5$ ohms. $L = 0.056$ henry. $C = 73\mu f$.

a previous paragraph. The boundary conditions are: $i = i_1$ at $t = 0$, and $q = 0$ at $t = 0$. i_1 will, of course, depend upon t_1 for its value.

EXERCISES

1. Prior to $t = 0$ a steady-state value of current is flowing through an E–R–L loop. At $t = 0$ the RL branch is disconnected from the battery and closed through a condenser branch.

(*a*) Derive the expression for current through the RLC circuit.

(*b*) Derive the expression for the condenser charge.

(*c*) If $E = 30$ volts, $I_0 = 2.5$ amperes, $L = 0.056$ henry, and $C = 140$ microfarads, calculate the length of time required for i to reach its first zero value. What is the frequency of oscillation of i about the zero line?

2. An elementary analysis of a neon-tube oscillator may be made by assuming that the tube has infinite resistance when not ionized and a very low resistance when it is ionized. The circuit arrangement is shown in Fig. 11. It will be assumed that the tube capacitance is negligibly small in comparison with C and that the tube ionizes on a rising voltage of e_1 (less than E) and de-ionizes at e_2 which is greater than zero but less than e_1.

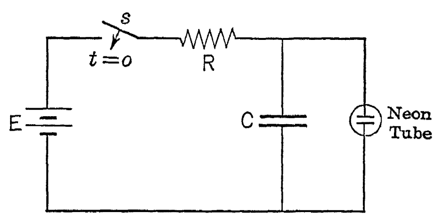


FIG. 11.—A simple neon-tube oscillator circuit.

Assuming that C possesses zero initial charge and that the switch s is closed at $t = 0$, analyze the resulting phenomena. State results in equation form.

3. Show that equation (41) reduces to:

$$i = \frac{E}{R\beta\sqrt{LC}}\epsilon^{-at}\sin(\beta t + \sigma)$$

where

$$\sigma = \tan^{-1}\frac{\beta}{a}$$

Make a graph of i vs. t for an arbitrarily assigned set of parameters.

4. Draw the circuit arrangement and describe the switching operations necessary to obtain a current variation similar to that shown in Oscillogram 10. (It may be of interest to know that Oscillogram 10 is the result of more than 100 superimposed light exposures, each of which is governed by the set of switching operations that is to be determined.)

5. A photographic record of the condenser discharge current in an RLC loop is shown in Oscillogram 11. The following information is furnished:

(*a*) The initial condenser voltage is equal to 110 volts.

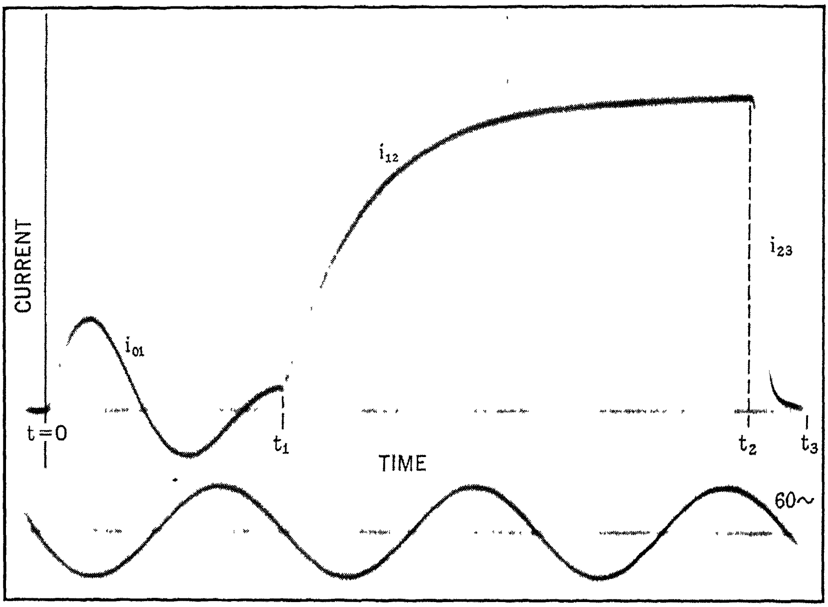
(*b*) The calibration of the current galvanometer is 0.6 ampere per unit length. Unit length is the distance between zero points on the timing wave.

(*c*) The timing wave is a 60-cycle-per-second variation.

What values of: R (in ohms), L (in henrys), and C (in microfarads) were employed in taking the oscillogram?

6. A relay has a resistance of 8 ohms and an inductance of 120 millihenrys. This relay and a 7-ohm resistor are connected in series across a 120-volt storage battery. A current of 6 amperes through the relay actuates a contactor which short-circuits the 7-ohm resistor. How long will it take, after the circuit is established, for the current to reach 90 per cent of its ultimate value? Assume constant self-inductance.

7. A relay having a resistance of 15 ohms and an inductance of 2.5 henrys is connected in series with a resistance of 10 ohms and a storage battery of 50 volts.[1]



Oscillogram 10.

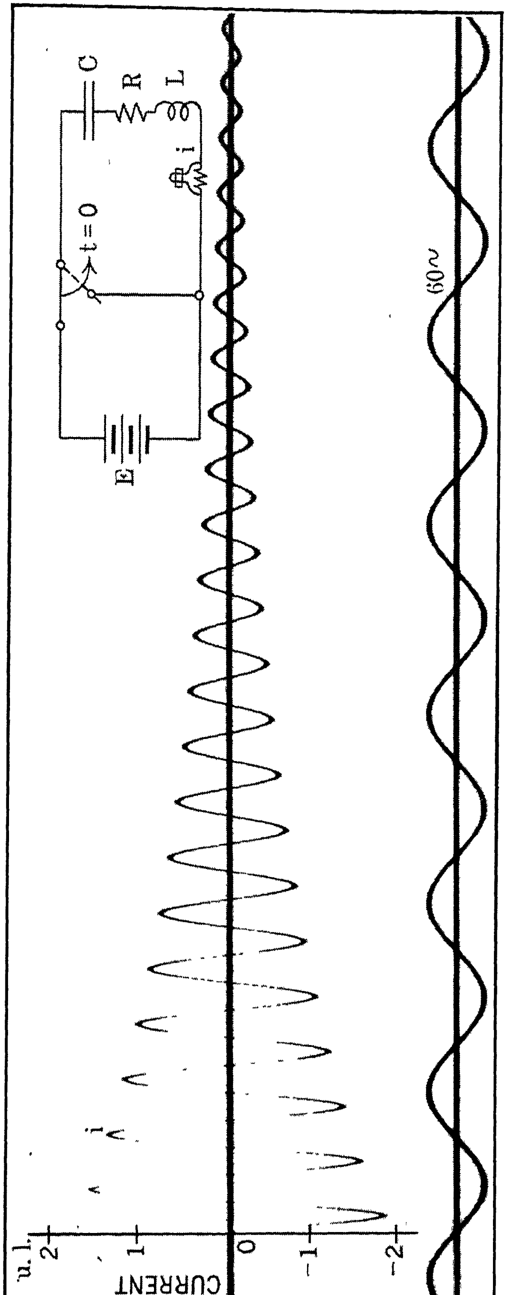
Compound transient to be used in connection with Exercise 4.

A current of 1.8 amperes through the coil actuates a contactor which shunts a 2-ohm non-inductive resistor across the winding of the relay. The contactor drops back when the relay current falls to 0.7 ampere. What time is required for the relay to go through a complete cycle? Assume constant self-inductance. Write the equations for: (*a*) The current through the relay when the contact is open. (*b*) The current through the relay when the contact is closed. (*c*) The current through the contactor circuit when the contact is closed.

Plot *a*, *b*, and *c* during the periods indicated.

8. A condenser of capacitance C farads which has been charged to a potential E_0 by a d-c voltage is allowed to discharge through a resistance and inductance. The

[1] This problem is generally similar to one given in Timbie and Bush, "Principles of Electrical Engineering," page 359, second edition, where a more detailed description of the relay action is presented.



OSCILLOGRAM 11.

An oscillatory condenser discharge current to be used in connection with Exercise 5.

discharge is oscillatory. Determine the energy resident in the inductance and capacitance at any time t seconds after the beginning of the discharge. Determine the logarithmic decrement of oscillation of the energy resident in the inductance and capacitance during discharge.

Plot the curve of energy resident in the inductance and capacitance when $E_0 = 1000$ volts, $R = 25$ ohms, $L = 100$ millihenrys, and $C = 10$ microfarads. Determine the decrement of oscillation.

9. A condenser of capacity C farads which has been charged through a circuit in which $R^2 < \frac{4L}{C}$ to a potential E_0 by a d-c voltage is allowed to discharge through the resistance and inductance. Determine how much of the stored energy is dissipated in the resistance from the beginning of the discharge to a time t_1 seconds thereafter.

10. An E–R–L loop is carrying a steady current of 50 amperes at $t = 0$. $R = 1.3$ ohms and $L = 1.2$ henrys. At $t = 0$ the applied voltage, E, is instantly replaced by a non-inductive resistance of 1.7 ohms. At what rate is the current decreasing when the energy remaining in the electromagnetic field is just equal to the total energy that has been dissipated since $t = 0$?

11. Refer to Fig. 3. The voltage $E = 120$ volts, $L = 52$ henrys, $R = 24$ ohms, $R_d = 20$ ohms. How long a time does it take after the switch s is opened before 95 per cent of the stored energy is dissipated in heat? (These data are from an early Edison dynamo.)

CHAPTER IV

SERIES-PARALLEL CIRCUITS

The elementary principles which have thus far been treated find a wide variety of application in series-parallel circuit analysis. In general, the solution of a series-parallel circuit problem depends upon the evaluation of the current in each individual branch. Provided that the current in each branch is treated as if it were a distinct dependent variable, Kirchhoff's current law may be applied independently as many times as there are junctions in the circuit less one. The remaining relationships between the currents are obtained by applying Kirchhoff's emf law the required number of times, that is, as many times as the circuit has individual branches less the number of times the current equation has been applied. If as many independent relationships have been established as there are individual currents, a solution is theoretically possible. Practically, however, only simple combinations may be treated in this manner. When more than two or three interrelated differential equations are involved some systematized procedure must be employed to effect a simultaneous solution.

PARALLEL BRANCHES

Mathematical Analysis.—Neglecting the internal resistance of the battery, the potential difference applied to the parallel branches shown

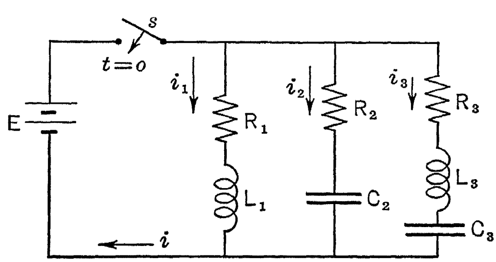


FIG. 1.—R_1L_1, R_2C_2, and $R_3L_3C_3$ in parallel.

in Fig. 1 is constant. Each branch may, under these conditions, be treated as an independent circuit. The expressions for the respective currents are:

$$i_1 = \frac{E}{R_1}\left(1 - \epsilon^{-\frac{R_1 t}{L_1}}\right) \tag{1}$$

$$i_2 = \frac{E}{R_2}\,\epsilon^{-\frac{t}{R_2 C_2}} \tag{2}$$

and

$$i_3 = \frac{E}{2bL_3}\,\epsilon^{-at}\left(\epsilon^{bt} - \epsilon^{-bt}\right) \tag{3}$$

where

$$a = \frac{R_3}{2L_3} \quad \text{and} \quad b = \sqrt{\frac{R_3^2}{4L_3^2} - \frac{1}{L_3 C_3}}$$

From Kirchhoff's current law it follows that

$$i = i_1 + i_2 + i_3 \tag{4}$$

Parallel combinations consisting of an RL branch and an RC branch are used in certain types of relays. The lagging nature of the RL branch is compensated by the impulsive nature of the RC branch with the result that a wide variety of effects may be produced. With respect to the terminals of the parallel combination the circuit may be made to simulate a pure resistance branch.

Considering only the R_1L_1 and the R_2C_2 branches of the circuit shown in Fig. 1, the resultant current becomes:

$$i = i_1 + i_2 = E\left\{\frac{1}{R_1} - \frac{1}{R_1}\,\epsilon^{-\frac{R_1 t}{L_1}} + \frac{1}{R_2}\,\epsilon^{-\frac{t}{R_2 C_2}}\right\} \tag{5}$$

Q_{20} is assumed to be equal to zero.

The necessary and sufficient conditions for equation (5) to be independent of time are:

$$R_1 = R_2 = \sqrt{\frac{L_1}{C_2}} \tag{6}$$

These are also the conditions necessary for the two branches to be in parallel resonance irrespective of the frequency of an applied alternating potential. When the above relationships between the four circuit parameters exist:

$$i = \frac{E}{R_1}$$

and the parallel combination reacts to an application of potential difference as would a single resistance branch. Graphs of i, i_1, and i_2 under the above conditions are shown in Fig. 2.

Oscillographic Verification.—By suitable adjustment of the parameters, the i graphs may be made to take various curvatures. Oscillogram 1 illustrates the variations of i, i_1, and i_2 for a set of parameters which approximate, quite closely, the conditions set forth in (6). The effect

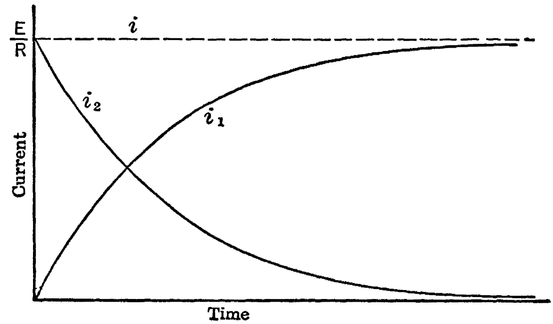


FIG. 2.—Current-time graphs. R_1L_1 in parallel with R_2C_2. $R_1 = R_2 = \sqrt{L_1/C_2}$.

of a somewhat greater departure from the conditions stated in (6) is shown in Oscillogram 2.

R IN SERIES WITH R_1L_1 AND R_2L_2 IN PARALLEL

Mathematical Development.—The obvious relationships between i, i_1, and i_2 of Fig. 3 are:

$$i = i_1 + i_2 \tag{7}$$

$$Ri + R_1 i_1 + L_1 \frac{di_1}{dt} = E \tag{8}$$

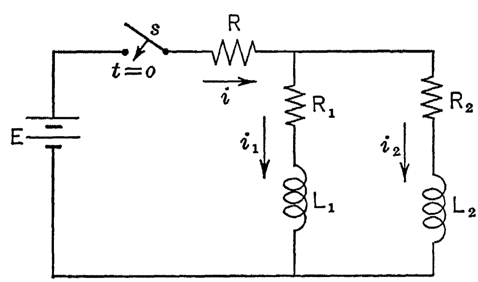
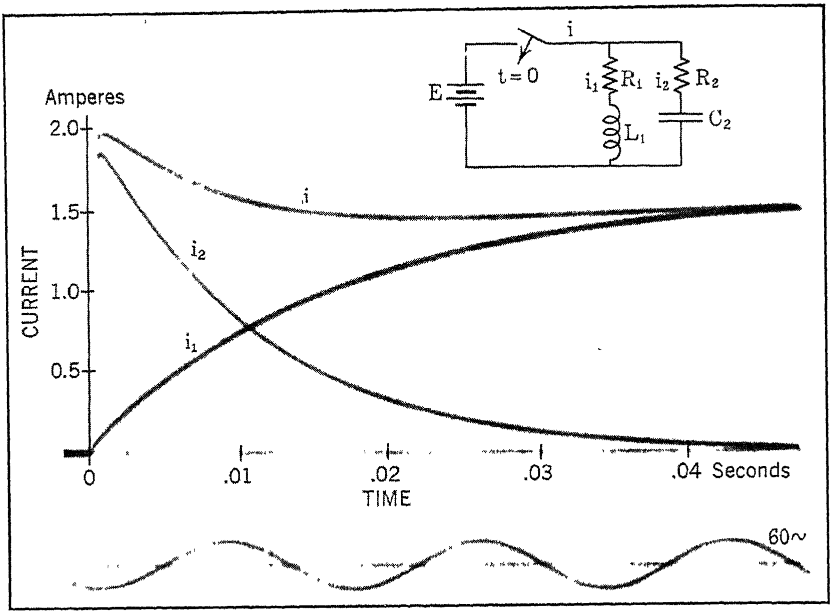


FIG. 3.—R in series with R_1L_1 and R_2L_2 in parallel.

$$Ri + R_2 i_2 + L_2 \frac{di_2}{dt} = E \tag{9}$$

and

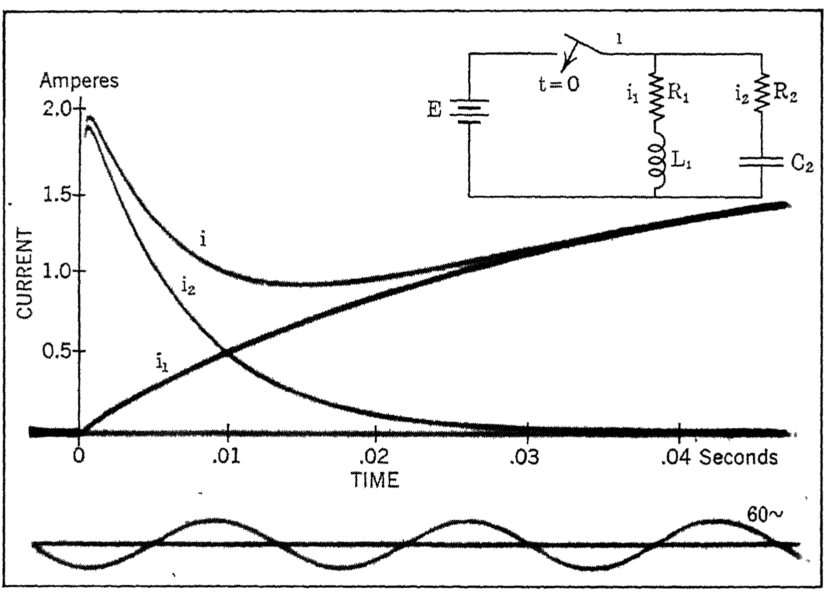
$$R_1 i_1 + L_1 \frac{di_1}{dt} - R_2 i_2 - L_2 \frac{di_2}{dt} = 0 \tag{10}$$



OSCILLOGRAM 1.

R_1L_1 in parallel with R_2C_2.

i_1 is the current in the R_1L_1 branch. i_2 is the current in the R_2C_2 branch. $i = i_1 + i_2$. $E = 55$ volts. $R_1 = 36$ ohms. $L_1 = 0.545$ henry. $R_2 = 28.5$ ohms. $C_2 = 390\mu$f.



OSCILLOGRAM 2.

R_1L_1 in parallel with R_2C_2.

i_1 is the current in the R_1L_1 branch. i_2 is the current in the R_2C_2 branch. $i = i_1 + i_2$.

Of the three emf equations, namely, (8), (9), and (10), it is evident that only two are independent.

The explicit expressions for the three currents may be obtained by the simultaneous solution of equations (7), (8), and (10).

$$R(i_1 + i_2) + R_1 i_1 + L_1 \frac{di_1}{dt} = E \tag{11}$$

$$Ri_2 + (R + R_1)i_1 + L_1 \frac{di_1}{dt} = E \tag{12}$$

Solving for i_2 in (12) gives:

$$i_2 = \frac{E}{R} - \frac{L_1}{R}\frac{di_1}{dt} - \left(\frac{R + R_1}{R}\right) i_1 \tag{13}$$

Differentiation yields:

$$\frac{di_2}{dt} = -\frac{L_1}{R}\frac{d^2 i_1}{dt^2} - \left(\frac{R + R_1}{R}\right)\frac{di_1}{dt} \tag{14}$$

Substituting now in equation (10):

$$R_1 i_1 + L_1 \frac{di_1}{dt} - R_2\left[\frac{E}{R} - \frac{L_1}{R}\frac{di_1}{dt} - \left(\frac{R + R_1}{R}\right) i_1\right]$$
$$- L_2\left[-\frac{L_1}{R}\frac{d^2 i_1}{dt^2} - \left(\frac{R + R_1}{R}\right)\frac{di_1}{dt}\right] = 0 \tag{15}$$

Rearrangement of the above yields:

$$\frac{d^2 i_1}{dt^2} + \left[\frac{(R + R_2)L_1 + (R + R_1)L_2}{L_1 L_2}\right]\frac{di_1}{dt}$$
$$+ \left[\frac{R_1 R + R_2 R + R_1 R_2}{L_1 L_2}\right] i_1 = \frac{ER_2}{L_1 L_2} \tag{16}$$

Abbreviating the above coefficients:

$$\frac{d^2 i_1}{dt^2} + J\frac{di_1}{dt} + K i_1 = \frac{ER_2}{L_1 L_2} \tag{17}$$

Breaking the current up into its two components:

$$i_1 = i_{1_s} + i_{1_t}$$

where

$$i_{1_s} = E\left[\frac{R_2}{R_1 R + R_2 R + R_1 R_2}\right] = \frac{E}{R_{1_{eq}}} \tag{18}$$

and i_{1_t} is the transient component of i_1.

Since $Ki_{1_s} = \frac{ER_2}{L_1 L_2}$, it follows that:

$$\frac{d^2 i_{1_t}}{dt^2} + J\frac{di_{1_t}}{dt} + Ki_{1_t} = 0 \tag{19}$$

$$i_{1_t} = A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \tag{20}$$

where $\alpha_1 \neq \alpha_2$.

α_1 and α_2 may be determined in the usual manner by substituting the above expression for i_{1_t} together with its first and second derivatives in equation (19).

$$A_1\epsilon^{\alpha_1 t}(\alpha_1{}^2 + J\alpha_1 + K) + A_2\epsilon^{\alpha_2 t}(\alpha_2{}^2 + J\alpha_2 + K) = 0 \tag{21}$$

from which

$$\alpha_1 = \frac{-J + \sqrt{J^2 - 4K}}{2} \tag{20a}$$

and

$$\alpha_2 = \frac{-J - \sqrt{J^2 - 4K}}{2} \tag{20b}$$

$$J = \frac{(R + R_2)L_1 + (R + R_1)L_2}{L_1 L_2}$$

and

$$K = \frac{R_1 R + R_2 R + R_1 R_2}{L_1 L_2}$$

It is simply a matter of algebra to show that J^2 is always equal to or greater than $4K$. Therefore α_1 and α_2 are real numbers, the magnitudes of which depend upon the values of R, R_1, R_2, L_1, and L_2.

The constants of integration, A_1 and A_2, are determined in the usual manner.

$$i_1 = \frac{E}{R_{1eq}} + A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \tag{22}$$

Since:

$$i_1 = 0 \text{ at } t = 0$$

$$A_1 + A_2 = -\frac{E}{R_{1eq}}$$

where

$$R_{1eq} = \frac{RR_1 + RR_2 + R_1 R_2}{R_2} \tag{23}$$

From equation (8):

$$\left(\frac{di_1}{dt}\right)_{t=0} = \frac{E}{L_1}$$

Therefore:

$$A_1\alpha_1 + A_2\alpha_2 = \frac{E}{L_1}$$

Substituting for A_2 gives:

$$A_1\alpha_1 - A_1\alpha_2 = \frac{E\alpha_2}{R_{1eq}} + \frac{E}{L_1}$$

from which:

$$A_1 = \frac{E}{\alpha_1 - \alpha_2}\left[\frac{1}{L_1} + \frac{\alpha_2}{R_{1eq}}\right] \tag{24}$$

$$A_2 = \frac{E}{\alpha_2 - \alpha_1}\left[\frac{1}{L_1} + \frac{\alpha_1}{R_{1eq}}\right] \tag{25}$$

The complete expression for i, as given by equation (22), therefore consists of the steady-state term, $\frac{E}{R_{1eq}}$, and two exponential terms. The coefficients of t in the exponential terms have been evaluated in terms of the circuit parameters, and the coefficients of $\epsilon^{\alpha_1 t}$ and $\epsilon^{\alpha_2 t}$ are given by equations (24) and (25) respectively. Thus i is completely defined in terms of the applied potential difference and the circuit parameters.

i_2 may now be evaluated in terms of i_1 and $\frac{di_1}{dt}$ in accordance with the relationship stated in (13). From the symmetry of equations (8) and (9), however, it is evident that i_2 takes the following form:

$$i_2 = \frac{E}{R_{2eq}} + B_1\epsilon^{\alpha_1 t} + B_2\epsilon^{\alpha_2 t} \tag{26}$$

where

$$R_{2eq} = \frac{RR_1 + RR_2 + R_1R_2}{R_1} \tag{27}$$

and

$$B_1 = \frac{E}{(\alpha_1 - \alpha_2)}\left[\frac{1}{L_2} + \frac{\alpha_2}{R_{2eq}}\right] \tag{28}$$

$$B_2 = \frac{E}{(\alpha_2 - \alpha_1)}\left[\frac{1}{L_2} + \frac{\alpha_1}{R_{2eq}}\right] \tag{29}$$

The current through R is the sum of i_1 and i_2 and may be expressed:

$$i = \frac{E}{R_{eq}} + (A_1 + B_1)\epsilon^{\alpha_1 t} + (A_2 + B_2)\epsilon^{\alpha_2 t} \tag{30}$$

where

$$R_{eq} = R + \frac{R_1 R_2}{R_1 + R_2}$$

Equations (22), (26), and (30) show that i_1, i_2, and i have the same general mathematical form. The steady-state terms are, of course, gov-

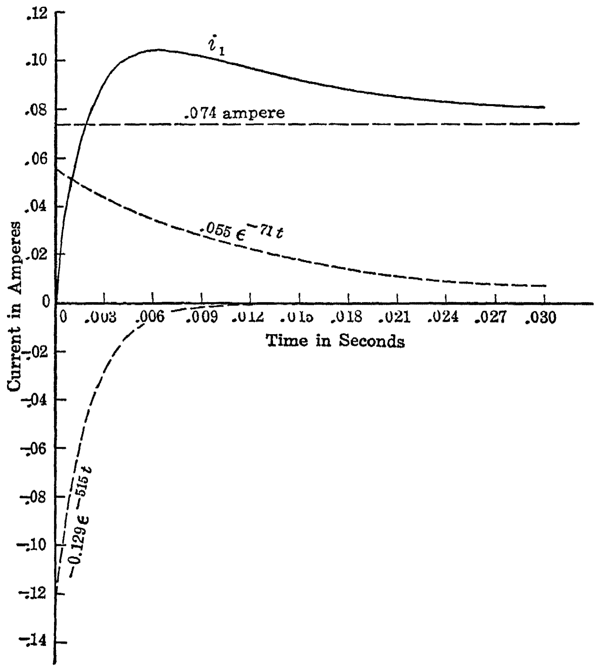


FIG. 4.—Plotted solution of equation (31).

erned entirely by the magnitudes of R_1, R_2, and R. The transient terms, governed as they are by the sum of two exponential components, are of a composite nature, the exact variation of which depends upon the relative values of L_1 and L_2 as well as the relative values of R_1, R_2, and R. A numerical case will serve to illustrate the nature of the component variations.

Numerical Example.—Let R = 3.83 ohms, R_1 = 3.76 ohms, R_2 = 2.42 ohms, L_1 = 0.016 henry, L_2 = 0.056 henry, E = 1 volt. With these particular parameters, the numerical magnitudes of the following quantities become:

$$\begin{aligned} J &= 586 & R_{1eq} &= 13.52 \\ K &= 36{,}500 & A_1 &= 0.055 \\ \alpha_1 &= -71 & A_2 &= -0.129 \\ \alpha_2 &= -515 & R_{2eq} &= 8.71 \\ \alpha_1 - \alpha_2 &= 444 & B_1 &= -0.0929 \\ \alpha_2 - \alpha_1 &= -444 & B_2 &= -0.0219 \end{aligned}$$

Assuming that unit potential difference is applied to the circuit at $t = 0$:

$$i_1 = 0.074 + 0.055\epsilon^{-71t} - 0.129\epsilon^{-515t} \tag{31}$$

A graphical solution of equation (31) is given in Fig. 4. By virtue of the positive sign of A_1 and the low value of α_1 as compared with α_2, the magnitude of i_1 becomes greater than its steady-state value, 0.074 ampere, during the early part of the transient period. A physical explanation of this phenomenon lies in the fact that during the early part of the transient period the voltage drop across R_1L_1 is greater than the steady-state voltage drop across the parallel branches. The R_2L_2 branch, being more highly inductive than the R_1L_1 branch, is rather effective in retarding the growth of the current. The result is that during the early period Ri is much smaller than it later becomes.

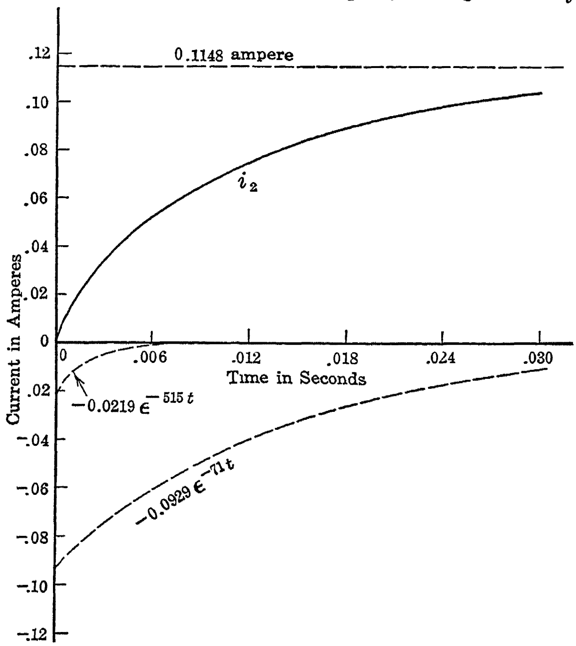
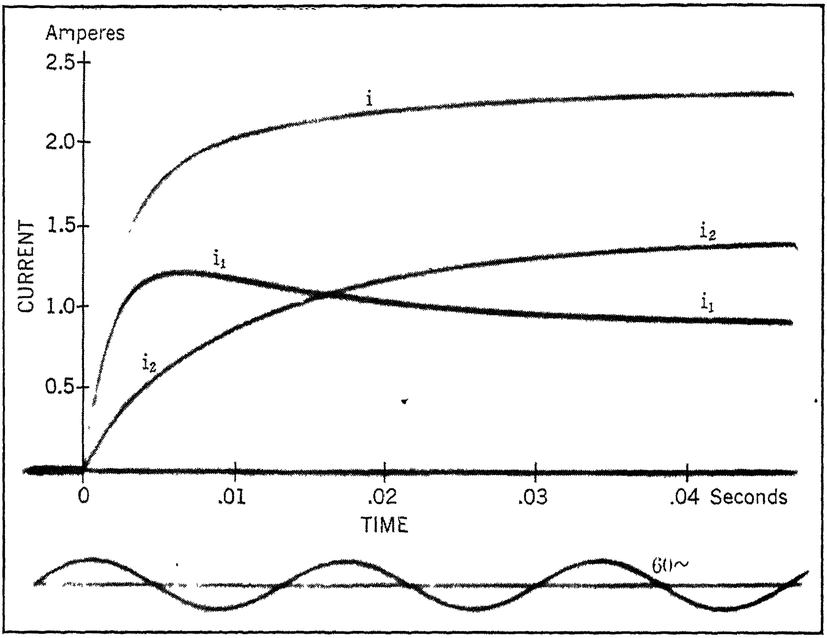


FIG. 5.—Plotted solution of equation (32).

$$i_2 = 0.1148 - 0.0929\epsilon^{-71t} - 0.0219\epsilon^{-515t} \tag{32}$$

Equation (32) is the expression for the current through the branch which possesses the higher ratio of L to R. The exponential terms which combine to form the transient component of the current are, in this case, both negative. i_2 does not attain values greater than its steady-state magnitude. A graphical representation of i_2 together with its component parts is shown in Fig. 5.

Oscillographic Verification.—The simultaneous variations of i, i_1, and i_2, for a set of parameters similar to those employed in the above numerical example, are illustrated by Oscillogram 3. i is, of course,



OSCILLOGRAM 3.
R in series with R_1L_1 and R_2L_2 in parallel. (See Fig. 3.)
i_1 is the current through R_1L_1. i_2 is the current through R_2L_2.
i is the current through R. $i = i_1 + i_2$.
$E = 12$ volts. $R = 3.83$ ohms. $R_1 = 3.76$ ohms. $R_2 = 2.42$ ohms.
$L_1 = 0.016$ henry. $L_2 = 0.056$ henry.

the sum of i_1 and i_2. The equal calibrations of the current galvanometers facilitate a quantitative study of the three simultaneous variations.

R IN SERIES WITH TWO IDENTICAL PARALLEL INDUCTIVE BRANCHES

Mathematical Procedure.—The expressions for current are greatly simplified in the series-parallel case heretofore considered when $R_1 = R_2$ and $L_1 = L_2$. Under these conditions:

$$\alpha_1 = -\frac{R_1}{L_1}$$

$$\alpha_2 = -\frac{(2R + R_1)}{L_1}$$

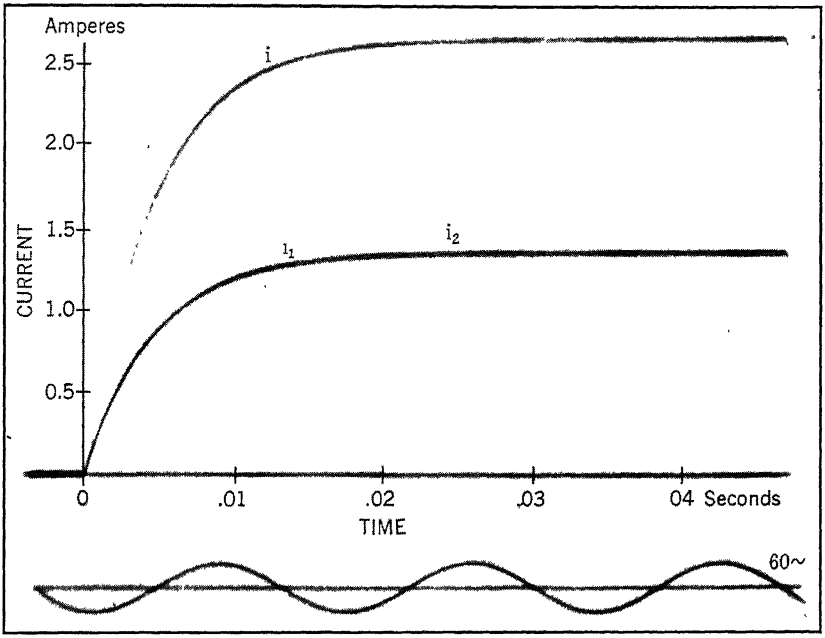
$$A_1 = B_1 = 0$$

$$A_2 = B_2 = -\frac{E}{(2R + R_1)}$$

$$R_{1_{eq}} = R_{2_{eq}} = (2R + R_1)$$

and

$$i_1 = i_2 = \frac{i}{2} = \frac{E}{(2R + R_1)}\left[1 - \epsilon^{-\frac{(2R+R_1)t}{L_1}}\right] \tag{33}$$



OSCILLOGRAM 4.

R in series with two identical parallel inductive branches.

$i_1 = i_2$; $i = i_1 + i_2$.

E = 16 volts. R = 4.2 ohms. $R_1 = R_2$ = 2.7 ohms. $L_1 = L_2$ = 0.056 henry.

Similar results may be obtained directly from equations (8) and (9). The transient component of i_1, i_2, or i consists of a single exponential term, the time constant of which is equivalent to that of a series circuit having $\left(R + \frac{R_1}{2}\right)$ ohms resistance and $L_1/2$ henrys inductance.

Oscillographic Demonstration.—Oscillogram 4 illustrates the three current variations for the particular case under discussion. Since $i_1 = i_2$, these two currents are shown superimposed on each other. i is shown to be equal to $2i_1$, throughout its entire variation.

R IN SERIES WITH R_1 AND C_2 IN PARALLEL

The input to many types of amplifiers takes the form of an Ri drop, and under certain conditions it becomes very desirable to know the transient response of the input circuit. In general, the series impedance and the shunt capacitance must be taken into consideration. If the resistances are very high as compared with the self-inductances the effects of the latter may be neglected in the first approximation. An example of such a case would be a photoelectric cell energizing a resistance which in turn forms the input circuit of an amplifier. The

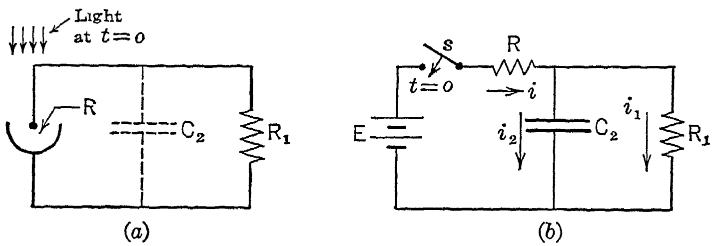


FIG. 6.—R in series with R_1 and C_2 in parallel.

photoelectric cell circuit shown in Fig. 6 (*a*) is approximately simulated by the circuit shown in Fig. 6 (*b*).

The response of the output circuit in either case is governed by R_1i_1, and the nature of i_1 is governed by the relative magnitudes of R, R_1, and C_2. The resistance of the cell, R, constitutes the major part of the series impedance and is usually very large as compared with R_1. The self-inductances of R and R_1 are considered to be negligibly small. If R_1i_1 is to respond as rapidly as possible to an application of potential difference, C must be kept as small as possible. The manner in which the various parameters effect the response is best obtained by evaluating i_1:

$$i = i_1 + i_2 \tag{34}$$

$$Ri + R_1i_1 = E \tag{35}$$

$$R_1i_1 - \frac{\int_0^t i_2dt}{C_2} = 0 \tag{36}$$

Q_{20} is assumed to be equal to zero.

Substituting for i in equation (35):

$$Ri_2 + (R + R_1)i_1 = E \tag{37}$$

From equation (36):

$$i_2 = R_1C_2\frac{di_1}{dt} \tag{38}$$

$$(R + R_1)i_1 + RR_1C_2\frac{di_1}{dt} = E \tag{39}$$

and

$$i_1 = \frac{E}{R + R_1}\left[1 - \epsilon^{-\frac{(R+R_1)}{RR_1C_2}t}\right] \tag{40}$$

The shunt capacitance C_2 is responsible for a retardation in the growth of i_1, and therefore causes the input to the amplifier, namely, the R_1i_1 drop, to lag the sudden application of potential difference. An ideal response would be independent of time, and to attain this end the coefficient of t in equation (40) is made as large as possible. With R assumed fixed, C_2 and R_1 would be kept as small as possible consistent with satisfactory operation. In the photoelectric driving circuit, C_2 represents the capacitance of the electrodes and connecting leads. The magnitude of R_1 is governed by two antagonistic factors. For maximum output of the photoelectric cell, circuit R_1 should be large, that is, comparable in magnitude to R. Since R is very large any attempt to match it results in a greater retardation of the response than is desirable. There are, of course, many cases where a lag in the response is not detrimental as in ordinary industrial applications. In certain television circuits, however, it is important that the lag be made as small as possible.

R IN SERIES WITH R_1L_1 AND R_2C_2 IN PARALLEL

The natural response of an R_1L_1 branch may be greatly changed by placing in parallel with it an R_2C_2 branch. Fig. 7 represents such a circuit combination. Including the inductive effect of the 1 branch and the resistive effect of the 2 branch adds materially to the variety of possible responses. The solutions of i_1, i_2, and i are outlined in operational form below. The substitution of p for d/dt and of $1/p$ for $\int^t dt$ systematizes the simultaneous solution as well as the evaluation of the constants of integration.

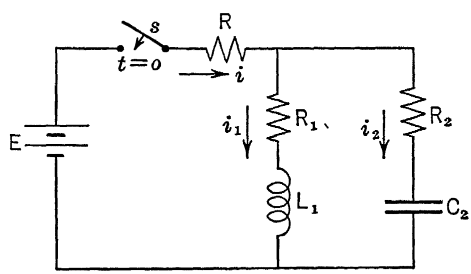


FIG. 7.—R in series with R_1L_1 and R_2C_2 in parallel.

The Operational Solution.—In operational form the fundamental relationships used are:

$$i = i_1 + i_2 \tag{41}$$

$$Ri + R_1 i_1 + L_1 p i_1 = E \tag{42}$$

and

$$R_1 i_1 + L_1 p i_1 - R_2 i_2 - \frac{i_2}{C_2 p} = 0 \tag{43}$$

THE i_1 SOLUTION.—Substituting for i in equation (42),

$$Ri_2 + (R + R_1)i_1 + L_1 p i_1 = E \tag{44}$$

Solving equations (43) and (44) simultaneously for i_1 yields:

$$i_1 = \frac{E(R_2C_2p + 1)}{(RL_1C_2 + R_2L_1C_2)p^2 + (RR_1C_2 + RR_2C_2 + R_1R_2C_2 + L_1)p + (R_1 + R)} \tag{45}$$

The functions of p that appear in the numerator and denominator of equation (38) must be interpreted if an actual solution is to be effected. Heaviside's expansion theorem offers a ready interpretation of such functions.[1]

When

$$i = \frac{EY(p)}{Z(p)}\mathbf{1},$$

$$i = E\left[\frac{Y(0)}{Z(0)} + \sum_{\substack{p=p_1\\p=p_2}} \frac{Y(p)\epsilon^{pt}}{p\dfrac{dZ(p)}{dp}}\right] \tag{46}$$

Adopting certain abbreviations:

$$i_1 = E\frac{(R_2C_2p + 1)}{Jp^2 + Kp + S}\mathbf{1} \tag{47}$$

where

$$J = (RL_1C_2 + R_2L_1C_2) \tag{48}$$

$$K = (RR_1C_2 + RR_2C_2 + R_1R_2C_2 + L_1) \tag{49}$$

$$S = (R_1 + R) \tag{50}$$

It will be observed that for any particular set of parameters, J, K, and S reduce to simple numerical values.

$$Y(p) = (R_2C_2p + 1)$$

[1] A proof of Heaviside's expansion theorem together with the exact meaning of the symbols is to be found in the Appendix.

and

$$Z(p) = Jp^2 + Kp + S$$

$$\frac{Y(0)}{Z(0)} = \frac{1}{S} = \frac{1}{R_1 + R}$$

The roots of $Z(p) = 0$ are:

$$p_1 = \frac{-K + \sqrt{K^2 - 4JS}}{2J}$$

$$p_2 = \frac{-K - \sqrt{K^2 - 4JS}}{2J}$$

$$\left[\frac{Y(p)}{p\,\dfrac{dZ(p)}{dp}}\right]_{\substack{\text{for } p=p_1\\ \text{or } p=p_2}} = \left[\frac{(R_2C_2p + 1)}{(2Jp^2 + Kp)}\right]_{\substack{\text{for } p=p_1\\ \text{or } p=p_2}}$$

Substituting the above values in equation (46) the expression for i_1 becomes:

$$i_1 = E\left[\frac{1}{(R_1 + R)} + A_1\epsilon^{p_1t} + A_2\epsilon^{p_2t}\right] \tag{51}$$

where

$$A_1 = \frac{(R_2C_2p_1 + 1)}{(2Jp_1{}^2 + Kp_1)} \qquad A_2 = \frac{(R_2C_2p_2 + 1)}{(2Jp_2{}^2 + Kp_2)}$$

THE i_2 SOLUTION.—Simultaneous solution of equations (43) and (44) for i_2 yields:

$$i_2 = \frac{E(L_1C_2p^2 + R_1C_2p)}{(RL_1C_2+R_2L_1C_2)p^2+(RR_1C_2+RR_2C_2+R_1R_2C_2+L_1)p+(R+R_1)} \tag{52}$$

Employing the abbreviations defined by equations (48), (49), and (50) reduces the expression for i_2 to:

$$i_2 = E\,\frac{(L_1C_2p^2 + R_1C_2p)}{Jp^2 + Kp + S}\mathbf{1} \tag{53}$$

$$Y(p) = (L_1C_2p^2 + R_1C_2p)$$

$$Z(p) = Jp^2 + Kp + S$$

$$\frac{Y(0)}{Z(0)} = 0$$

Inasmuch as $Z(p)$ in equation (53) is equal to $Z(p)$ in equation (47), the roots of $Z(p) = 0$ will be the same, namely:

$$p_1 = \frac{-K + \sqrt{K^2 - 4JS}}{2J}$$

$$p_2 = \frac{-K - \sqrt{K^2 - 4JS}}{2J}$$

$$\left[\frac{Y(p)}{p\dfrac{dZ(p)}{dp}}\right] = \frac{(L_1C_2p + R_1C_2)}{(2Jp + K)}$$

The complete expression for i_2 now becomes:

$$i_2 = E[B_1\epsilon^{p_1t} + B_2\epsilon^{p_2t}] \tag{54}$$

where

$$B_1 = \frac{(L_1C_2p_1 + R_1C_2)}{(2Jp_1 + K)}$$

$$B_2 = \frac{(L_1C_2p_2 + R_1C_2)}{(2Jp_2 + K)}$$

The i Solution.—i may now be expressed in terms of the foregoing solutions for i_1 and i_2, thus:

$$i = E\left[\frac{1}{(R_1 + R)} + (A_1 + B_1)\epsilon^{p_1t} + (A_2 + B_2)\epsilon^{p_2t}\right] \tag{56}$$

A general interpretation of equations (51), (54), and (56) is suggested as a classroom exercise. The solution of i_1, i_2, and i by a more conventional method and a correlation of the results with equations (51), (54), and (56) will prove to be a worthwhile procedure for those not versed in operational methods.

RL IN SERIES WITH R_1L_1 AND $R_2L_2C_2$ IN PARALLEL

General Method of Solution.—Although the series-parallel circuit represented by Fig. 8 is far from being a perfectly general case, it is sufficiently complex to illustrate certain limitations of mathematical predetermination. The fundamental relationships are:

$$i_1 + i_2 = i \tag{57}$$

$$Ri + L\frac{di}{dt} + R_1i_1 + L_1\frac{di_1}{dt} = E \tag{58}$$

$$Ri + L\frac{di}{dt} + R_2i_2 + L_2\frac{di_2}{dt} + \frac{q_2}{C_2} = E \tag{59}$$

$$R_1i_1 + L_1\frac{di_1}{dt} - R_2i_2 - L_2\frac{di_2}{dt} - \frac{q_2}{C_2} = 0 \tag{60}$$

Substituting the value of i as given by equation (57) in equation (58) gives:

$$Ri_2 + L\frac{di_2}{dt} + (R + R_1)i_1 + (L + L_1)\frac{di_1}{dt} = E \tag{61}$$

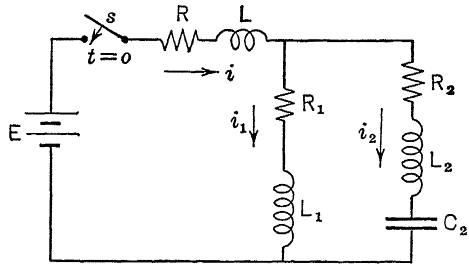


FIG. 8.—RL in series with R_1L_1 and $R_2L_2C_2$ in parallel.

If equations (60) and (61) are made symmetrical by successive differentiation, simultaneous solutions of i_1 and i_2 may be obtained. Differentiating and rearranging equations (60) and (61) yields:

$$L_1\frac{d^3i_1}{dt^3} + R_1\frac{d^2i_1}{dt^2} - L_2\frac{d^3i_2}{dt^3} - R_2\frac{d^2i_2}{dt^2} - \frac{1}{C_2}\frac{di_2}{dt} = 0 \tag{62}$$

and

$$(L + L_1)\frac{d^3i_1}{dt^3} + (R + R_1)\frac{d^2i_1}{dt^2} + L\frac{d^3i_2}{dt^3} + R\frac{d^2i_2}{dt^2} = 0 \tag{63}$$

The steady-state values of i_1 and i_2 are known to be $\dfrac{E}{R + R_1}$ and zero respectively. This information together with the form of equations (62) and (63) is sufficient to establish generalized expressions for i_1 and i_2.

$$i_1 = \frac{E}{R + R_1} + A_1\epsilon^{\alpha_1t} + A_2\epsilon^{\alpha_2t} + A_3\epsilon^{\alpha_3t} \tag{64}$$

$$i_2 = 0 + B_1\epsilon^{\alpha_1t} + B_2\epsilon^{\alpha_2t} + B_3\epsilon^{\alpha_3t} \tag{65}$$

The above expressions might, from a mathematical point of view, be classed as solutions of i_1 and i_2, but they do not yield the information which is necessary in order that they be classed as physical solutions.

For a particular set of circuit parameters, α_1, α_2, and α_3 may be evaluated by detailed computation, and from their values certain pertinent information as to the current variations may be derived. The exact nature of the variations, however, cannot be predicted until the values of A_1, A_2, A_3, B_1, B_2, and B_3 have been determined. If ordinary methods are employed the labor involved in such determination is prohibitive. The labor may be reduced to a minimum by employing particular devices, amongst which are determinants and the Heaviside method.

The Operational Solution.—A detailed treatment of the problem at hand will serve to illustrate some of the limitations of the device known as Heaviside's expansion theorem. The forms of the emf equation are somewhat more simple when expressed operationally.

$$(R + Lp)i + (R_1 + L_1p)i_1 = E \tag{58a}$$

$$(R + Lp)i + \left(R_2 + L_2p + \frac{1}{C_2p}\right) i_2 = E \tag{59a}$$

$$(R_1 + L_1p)i_1 - \left(R_2 + L_2p + \frac{1}{C_2p}\right) i_2 = 0 \tag{60a}$$

Explicit expressions for i_1, i_2, and i are now obtainable. Substituting the value of i as given by equation (57) in equation (58a) gives:

$$[R + Lp]i_2 + [R_1 + R + (L_1 + L)p]i_1 = E \tag{66}$$

From equation (60a):

$$i_2 = \frac{(R_1 + L_1p)i_1}{\left(R_2 + L_2p + \dfrac{1}{C_2p}\right)} \tag{67}$$

Substituting the above value of i_2 in equation (66) yields,

$$\frac{(R_1 + L_1p)(R + Lp)}{\left(R_2 + L_2p + \dfrac{1}{C_2p}\right)} i_1 + [(R_1 + R) + (L_1 + L)p]i_1 = E \tag{68}$$

from which:

$$i_1 = \frac{E\left(R_2 + L_2p + \dfrac{1}{C_2p}\right)}{\left\{(R_1 + L_1p)(R + Lp) + [R_1 + R + (L_1 + L)p]\left[R_2 + L_2p + \dfrac{1}{C_2p}\right]\right\}} \tag{69}$$

$$= \frac{E\left(L_2p^2+R_2p+\dfrac{1}{C_2}\right)}{\left\{\begin{matrix}(L_1L+L_2L+L_1L_2)p^3+(LR_1+L_1R+L_2R_1+L_2R+L_1R_2+LR_2)p^2 \\ +\left[R_1R+R_2R+R_1R_2+\dfrac{(L_1+L)}{C_2}\right]p+\dfrac{R_1+R}{C_2}\end{matrix}\right\}} \tag{70}$$

The explicit expression for i_2 may be obtained in a similar manner and is:

$$i_2 = \frac{E(L_1p^2+R_1p)}{\left\{\begin{matrix}(L_1L+L_2L+L_1L_2)p^3+(LR_1+L_1R+L_2R_1+L_2R+L_1R_2+LR_2)p^2 \\ +\left[R_1R+R_2R+R_1R_2+\dfrac{(L_1+L)}{C_2}\right]p+\dfrac{R_1+R}{C_2}\end{matrix}\right\}} \tag{71}$$

For the sake of simplicity in writing, let the following abbreviation be adopted:

$$J = (L_1L + L_2L + L_1L_2) \tag{72}$$

$$K = (LR_1 + L_1R + L_2R_1 + L_2R + L_1R_2 + LR_2) \tag{73}$$

$$N = \left(R_1R + R_2R + R_1R_2 + \frac{L_1 + L}{C_2}\right) \tag{74}$$

$$S = \frac{(R + R_1)}{C_2} \tag{75}$$

Equations (70) and (71) then become:

$$i_1 = E\frac{L_2p^2 + R_2p + \dfrac{1}{C_2}}{(Jp^3 + Kp^2 + Np + S)}\mathbf{1} \tag{70a}$$

$$i_2 = E\frac{L_1p^2 + R_1p}{(Jp^3 + Kp^2 + Np + S)}\mathbf{1} \tag{71a}$$

i_1 and i_2 are expressed as functions of p which in turn may be interpreted by the expansion theorem.

$$i_1 = E\left\{\frac{Y(0)}{Z(0)} + \frac{Y(p_1)}{p_1\left[\dfrac{dZ(p)}{dp}\right]_{p=p_1}}\epsilon^{p_1t} + \frac{Y(p_2)}{p_2\left[\dfrac{dZ(p)}{dp}\right]_{p=p_2}}\epsilon^{p_2t} + \frac{Y(p_3)}{p_3\left[\dfrac{dZ(p)}{dp}\right]_{p=p_3}}\epsilon^{p_3t}\right\} \tag{76}$$

where p_1, p_2, p_3 are the roots of

$$Z(p) = 0$$

Inasmuch as $Z(p)$ is a third-degree expression, the coefficients of which are cumbersome functions of the circuit parameters, it is evident that equation (76) is quite as far from a physical solution as was equation (64). However, the present form of the expression for i_1 lends itself to numerical solution somewhat more readily than equation (64), and for that reason it will be employed in the numerical example that follows.

Numerical Example.—Let $R = 1.0$ ohm, $L = 0.011$ henry, $R_1 = 17.7$ ohms $L_1 = 0.112$ henry, $R_2 = 5.0$ ohms, $L_2 = 0.056$ henry, and $C_2 = 28.8$ microfarads. Then:

$$J = (L_1L + L_2L + L_1L_2) = 0.008118$$

$$K = L(R_1 + R_2) + L_1(R + R_2) + L_2(R + R_1) = 1.969$$

$$N = \left(RR_1 + RR_2 + R_1R_2 + \frac{L_1 + L}{C_2}\right) = 4381$$

$$S = \frac{R + R_1}{C_2} = 649300$$

$$Z(p) = Jp^3 + Kp^2 + Np + S$$

$$= 0.008118p^3 + 1.969p^2 + 4381p + 649300$$

$Z(p) = 0$ may be written, to slide-rule accuracy, as follows:

$$p^3 + 242p^2 + 0.539 \times 10^6 p + 79.8 \times 10^6 = 0 \tag{77}$$

Any of the various methods [1] for the evaluation of roots of algebraic equations may be employed in determining the roots of equation (77). The roots are:

$$p_1 = -45 + j725 \tag{78}$$

$$p_2 = -45 - j725 \tag{79}$$

$$p_3 = -152 \tag{80}$$

These values may be checked by writing $Z(p)$ in factored style and performing the indicated multiplications.

The conjugate roots, p_1 and p_2, indicate the existence of an oscillatory component in the transient current. This is, of course, one of the modes of variation that is to be expected, but it is not until the roots of $Z(p) = 0$ have been evaluated that such is clearly evident. Since the operational expressions for i_1 and i_2 contain the same $Z(p)$, both variations of current will possess an oscillatory component. A consideration of the physical set-up (Fig. 8) will reveal the general character of i_1 and i_2. i_1 will rise to its $\dfrac{E}{R + R_1}$ value at a rate which is governed largely, but not wholly, by the ratio of $(R + R_1)$ to $(L + L_1)$. An oscillatory component that is governed to some extent by the values of R_2, L_2, and C_2 will be present during the transient

[1] The Graeffe or "root-squaring" method is described in the Appendix, and the evaluation of the roots of equation (77) is shown in detail on pages 303 and 304.

period. i_2 will consist of three exponential terms, two of which combine to form a damped oscillation.

Evaluation of $\left[\dfrac{Y(p)}{p\dfrac{dZ(p)}{dp}}\right]$ for p_1, p_2, and p_3 reveals the relative magnitudes of the component parts of i_2.

$$i_2 = E\left\{\frac{Y(0)}{Z(0)} + \frac{Y(p_1)}{p_1\left[\dfrac{dZ(p)}{dp}\right]_{p=p_1}}\epsilon^{p_1t} + \frac{Y(p_2)}{p_2\left[\dfrac{dZ(p)}{dp}\right]_{p=p_2}}\epsilon^{p_2t} + \frac{Y(p_3)}{p_3\left[\dfrac{dZ(p)}{dp}\right]_{p=p_3}}\epsilon^{p_3t}\right\} \tag{81}$$

From equation (71a):

$$Y(p) = L_1p^2 + R_1p$$

$$Z(p) = Jp^3 + Kp^2 + Np + S$$

The steady-state component of i_2 is:

$$\frac{Y(0)}{Z(0)} = 0$$

The coefficients of the exponential terms are:

$$\left[\frac{Y(p)}{p\dfrac{dZ(p)}{dp}}\right] = \frac{L_1p + R_1}{3Jp^2 + 2Kp + N}$$

Numerically:

$$\frac{Y(p_1)}{p_1\left[\dfrac{dZ(p)}{dp}\right]_{p=p_1}} = \frac{12\ 7 + j81.3}{-8540 + j1275} = \frac{82\ 3\epsilon^{j81^\circ\ 20'}}{8640\epsilon^{-j171^\circ\ 30'}} = 0\ 0095\epsilon^{-j90^\circ\ 10'}$$

$$\frac{Y(p_2)}{p_2\left[\dfrac{dZ(p)}{dp}\right]_{p=p_2}} = \frac{12\ 7 - j81\ 3}{-8540 - j1275} = \frac{82\ 3\epsilon^{-j81^\circ\ 20'}}{8640\epsilon^{-j171^\circ\ 30'}} = 0\ 0095\epsilon^{+j90^\circ\ 10'}$$

$$\frac{Y(p_3)}{p_3\left[\dfrac{dZ(p)}{dp}\right]_{p=p_3}} = 0.000152$$

Substituting the above values together with the values of p_1, p_2, and p_3 in equation (81),

$$i_2 = E[0\ 0095\epsilon^{-j90^\circ\ 10'}\epsilon^{(-45+j725)t} + 0\ 0095\epsilon^{+j90^\circ\ 10'}\epsilon^{(-45-j725)t} + 0.000152\epsilon^{-152t}]$$

$$i_2 = E[0.019\epsilon^{-45t}\cos(725t - 90^\circ\ 10') + 0\ 000152\epsilon^{-152t}] \tag{82}$$

$$i_2 = E[0.019\epsilon^{-45t}\sin 725t + 0\ 000152\epsilon^{-152t}],\ \text{approximately} \tag{82a}$$

The coefficient of the oscillatory term is more than one hundred times as large as that of the exponential term. It is further observed that the oscillatory component is damped by the factor ϵ^{-45t} while the remaining component is damped by the factor ϵ^{-152t}. Therefore the oscillatory term will be of greater magnitude and persist as

an appreciable quantity much longer than the single exponential term. The frequency of the oscillatory component is defined by the angular velocity, 725 radians per second, and is 115.3 cycles per second. Under the condition of unit voltage applied the magnitude of i_2 is expressed by the bracket term of equation (82) as two distinct functions of time.

The equation for i_1 may be determined in a manner similar to the one given above for i_2. In terms of numerical coefficients:

$$i_1 = E[0.0534 - 0.0535\epsilon^{-152t} - 0.00165\epsilon^{-45t}\cos(725t + 94° 45')] \quad (83)$$

in the case of i_1, a steady-state term is present and is indicated by the first term within the brackets in equation (83). It will be observed that the first two terms within the brackets correspond very closely to the expression for the rise of current in an

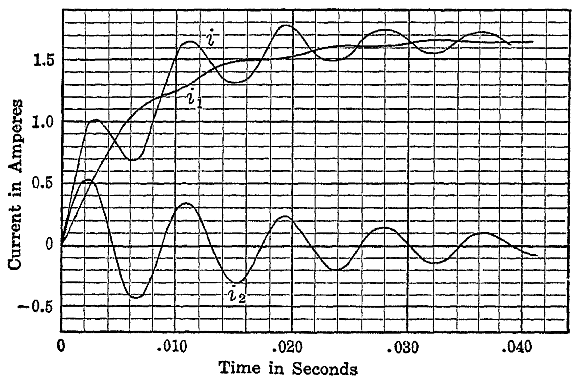
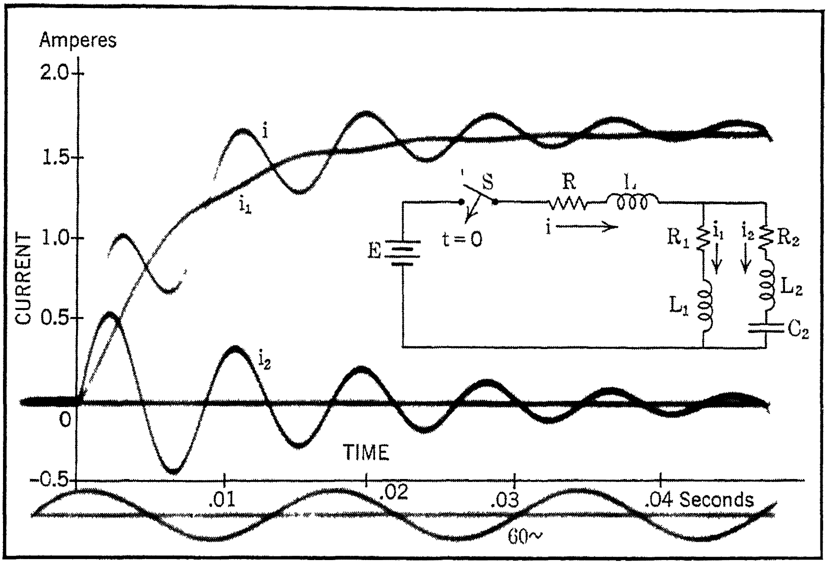


FIG. 9.—Plotted solutions of equations (82) and (83).

inductive branch. The oscillatory term is small compared with the other factors and as such does not influence the current variation materially.

Graphs of i_1, i_2 and i are shown in Fig. 9 for the case of $E = 30.5$ volts. The graphs of i_1 and i_2 are the plotted solutions of equations (82) and (83) respectively. i is, of course, the sum of i_1 and i_2. The large number of computations involved places a serious limitation upon numerical solutions, and for that reason oscillographic solutions are to be preferred when the circuit under consideration possesses more than three or four individual branches.

Oscillographic Verification.—An oscillographic solution of the present problem is given in Oscillogram 5, where the variations of i_1, i_2, and i are shown throughout the major portion of the transient period. The parameters employed correspond to those used in the above numerical example. Several features of the solution are clearly illustrated while others are scarcely discernible because of their relatively small magnitudes. The i_1, i_2, and i variations shown in Oscillogram 5 correspond very nicely with the numerically determined current-time graphs shown in Fig. 9.



OSCILLOGRAM 5.

RL in series with R_1L_1 and $R_2L_2C_2$ in parallel.

i_1 is the current through the R_1L_1 branch. i_2 is the current through the $R_2L_2C_2$ branch.

$i = i_1 + i_2$, the current through RL.

$E = 30.5$ volts. $R = 1.0$ ohm. $L = 0.011$ henry. $R_1 = 17.7$ ohms.

$L_1 = 0.112$ henry. $R_2 = 5.0$ ohms. $L_2 = 0.056$ henry. $C_2 = 28.8\mu$f.

EXERCISES

1. (*a*) Determine the natural response of the circuit shown in Fig. 7 by the ordinary differential equation method.

(*b*) Compare the results obtained with equations (51), (54), and (56).

(*c*) Interpret the results from a physical point of view as far as it is possible to do so with the equations in generalized form.

(*d*) Having selected a set of numerical parameters at pleasure, determine the numerical expressions for i_1, i_2, and i.

2. Prove that equations (64) and (65) are general solutions of equations (62) and (63).

3. State the relationships from which A_1, A_2, A_3, B_1, B_2, and B_3 of equations (64) and (65) may be determined.

4. Laboratory equipment permitting, make an oscillographic study of the circuit shown in Fig. 7. In lieu of an oscillographic study construct the current-time graphs of i_1, i_2, and i for an arbitrarily assigned set of circuit parameters.

5. Refer to Oscillogram 2. It is known that the voltage applied to the branches at $t = 0$ is 30.5 volts and that $R_1 = 17.5$ ohms. Find the values of L_1, R_2, and C_2. NOTE: The E/R_1 value of i_1 is not shown on the oscillogram.

CHAPTER V

INDUCTIVELY COUPLED CIRCUITS

Mutual Induction.—The studies which have thus far been made carried with them the tacit assumption that none of the magnetic flux established by the circuit under consideration linked with any neighboring circuit. In the case of series-parallel circuits it was further assumed that no magnetic coupling existed between the individual branches. The transient response of many electrical circuits is greatly influenced by the amount of magnetic coupling that is present, and the operation of many devices depends almost solely upon the phenomenon known as mutual induction. The present chapter devotes itself to simple circuit combinations in which mutual induction is an appreciable factor in governing the transient response.

Transient effects due to magnetic coupling are most easily analyzed in terms of the coefficient of mutual inductance M. M appears in the circuit equations as a fourth parameter even though it possesses the same dimensional qualities as the coefficient of self-inductance. M helps to govern the interaction between circuits whereas L affects only the reaction within the circuit itself.

The coefficient of mutual inductance between two circuits may be defined in terms of the change of flux linkage in one of the circuits per unit change of current in the other. The equivalent definition in terms of induced voltage is somewhat more applicable if M is constant. The mutual inductance of circuit 1 to circuit 2 is:

$$M_{12} = \frac{\Delta(N_2 \phi_{12})}{\Delta I_1}$$ Change in flux linkage of circuit 2 per unit change of current in circuit 1.

Or:

$$M_{12} = -\frac{e_2}{di_1/dt}$$ Voltage induced in circuit 2 by a unit rate of change of current in circuit 1.

Similarly, the mutual inductance of circuit 2 to circuit 1 is:

$$M_{21} = \frac{\Delta(N_1 \phi_{21})}{\Delta I_2}$$ Change in flux linkage of circuit 1 per unit change of current in circuit 2.

Or:

$$M_{21} = -\frac{e_1}{di_2/dt}$$ Voltage induced in circuit 1 by a unit rate of change of current in circuit 2.

If no magnetic material is present and the two circuits maintain fixed positions with respect to one another, $M_{12} = M_{21} = M$. Under these conditions the reluctance of the mutual flux path ($\mathcal{R}_{12}$ or $\mathcal{R}_{21}$) is a fixed quantity and $\mathcal{R}_{12}$ may be considered equal to $\mathcal{R}_{21}$. Thus:

$$M_{12} = \frac{N_2\phi_{12}}{I_1} = \frac{4\pi N_2 N_1}{\mathcal{R}_{12}}$$

and

$$M_{21} = \frac{N_1\phi_{21}}{I_2} = \frac{4\pi N_1 N_2}{\mathcal{R}_{21}}$$

If magnetic material is present $\mathcal{R}_{12}$ will not be equal to $\mathcal{R}_{21}$ unless the two electrical circuits are of exactly the same size and shape and are symmetrically located with respect to the magnetic material. Unless otherwise stated an absence of magnetic material will be assumed, in which case $M_{12} = M_{21} = M$.

THE $R_1L_1M\ R_2L_2$ COMBINATION

Physical Considerations.—Two RL circuits inductively coupled as shown in Fig. 1 will be known as the $R_1L_1M\ R_2L_2$ combination.

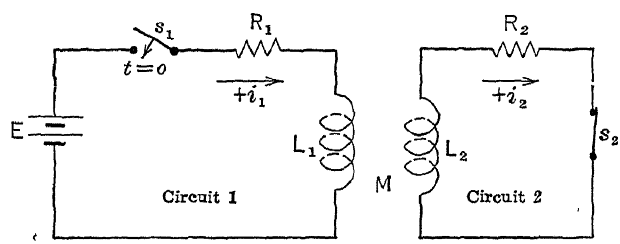


FIG. 1.—The $R_1L_1M\ R_2L_2$ combination.

Subscripts 1 refer to the driving or primary circuit, and subscripts 2 to the secondary circuit.

If the current in either circuit 1 or circuit 2 is to be determined the interaction between the two circuits must be considered. L_1 represents the total self-inductance of circuit 1, and L_2 that of circuit 2. L_1 may be defined as the flux linkage per unit current of circuit 1 when not influenced by circuit 2. Likewise L_2 is the flux linkage per unit current of circuit 2 when not influenced by circuit 1. In Fig. 1, it is assumed that a fractional part of the magnetic field that is established by i_1 links with circuit 2. It naturally follows that a fractional part of the magnetic field established by i_2 links with circuit 1. The effect of one circuit upon the other may be determined by taking into account the

voltages that are induced in these circuits as a result of variations in the mutual flux.

It is desirable at the outset to adopt some scheme for determining the correct sign of the mutually induced voltages. Certain of the more advanced problems require a strict adherence to such a procedure, especially in writing the fundamental equations for dynamic equilibrium.

The voltage induced in circuit 2 as a result of change in current in circuit 1 may be written:

$$e_{21} = -M\frac{di_1}{dt} \tag{1}$$

M is the flux linkage of circuit 2 per unit current in circuit 1. The minus sign affirms that e_{21} acts in a direction to produce a negative i_2. In the case of separate circuits the sign of i_2 must be determined with respect to the direction of the magnetic field established by i_2 rather than by any arbitrary direction around the circuit. A positive i_2 establishes flux through the coupled turns of circuit 1 in the same direction that a positive i_1 establishes flux in those same turns. The geometrical arrangement of the circuits and the mode of winding will determine the positive direction of current in circuit 2. The positive direction of i_1 is determined by the polarity of the voltage applied to circuit 1.

The voltage induced in circuit 1 as a result of change in current in circuit 2 may be written:

$$e_{12} = -M\frac{di_2}{dt} \tag{2}$$

In this case M is the number of flux linkages of circuit 1 per unit current in circuit 2. It is equal to the M in equation (1) when no magnetic material is present.

Adopting the above convention for the signs of the mutually induced voltages, it is a simple matter to write e_{12} and e_{21} into the equations for dynamic equilibrium. As induced or generated voltages they are negative quantities; as counter-voltages they appear in the left-hand member of the equations as positive counter-voltages.

Applying Kirchhoff's emf law to circuits 1 and 2, the following equations are obtained:

$$R_1i_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = E \tag{3}$$

and

$$R_2i_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} = 0 \tag{4}$$

From physical considerations, i_1 will be expected to increase in magnitude, after s_1 is closed, until its E/R_1 value is reached. The initial current in either circuit is zero at $t = 0$. The steady-state value of current in circuit 1 is, obviously, E/R_1, and the steady-state value of current in circuit 2 is, of course, equal to zero. As long as i_1 is increasing in magnitude the change in flux linkage produced by circuit 1 in circuit 2 will be such as to cause a negative i_2 to exist in circuit 2. It is evident that i_2 increases to a certain maximum value negatively and then recedes to zero according to some function of time.

Inspection of equation (3) shows that the rise in i_1 from zero to its steady-state value is influenced by the $M\dfrac{di_2}{dt}$ term. From $t = 0$ until i_2 reaches its negative maximum $\dfrac{di_2}{dt}$ will be negative and the $M\dfrac{di_2}{dt}$ term is a negative counter-voltage. As such it acts oppositely to the $L_1\dfrac{di_1}{dt}$ term. The effect of the mutual inductance during this period is therefore to lessen the effect of L_1. It will be remembered that the effect of L in a simple series circuit is to retard the change of current in that circuit and thus increase the time required for the current to reach its final or steady-state value. Thus, it is plain that the effect of M, for a period of time immediately following the closing of the switch, is to accelerate the building up of i_1. But after i_2 has reached its negative maximum $\dfrac{di_2}{dt}$ becomes positive and the $M\dfrac{di_2}{dt}$ term in equation (3) becomes a positive counter-voltage. As such it will retard the further building-up process of i_1. Viewed from a flux linkage standpoint the building up of i_2 to its negative maximum produces in circuit 1 a change of flux linkage such as to produce a positive i_1. The recession of i_2 from its negative maximum to its zero value produces a change of flux linkage with circuit 1 such as to produce a negative i_1.

The $M\dfrac{di_1}{dt}$ term, though written into equation (4) as a counter-voltage, may be considered the forcing voltage of circuit 2, which of course it is. Written on the right-hand side of the equation it appears as a negative driving voltage that gives rise to the negative i_2. The $M\dfrac{di_1}{dt}$ term becomes zero when i_1 has reached its steady-state value, so necessarily i_2 will also be zero at that time.

The effects of the coupling between circuit 1 and circuit 2 will become

more obvious after the mathematical expressions for i_1 and i_2 have been determined.

Mathematical Development.—The explicit expressions for i_1 and i_2 may be found by solving equations (3) and (4) simultaneously. For convenience these equations are reproduced here.

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = E \tag{3}$$

and

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = 0 \tag{4}$$

In order to obtain expressions that are in terms of a single dependent variable the following procedure may be employed. Differentiating equation (4):

$$R_2 \frac{di_2}{dt} + L_2 \frac{d^2 i_2}{dt^2} + M \frac{d^2 i_1}{dt^2} = 0 \tag{5}$$

From equation (3), it follows that:

$$\frac{di_2}{dt} = \frac{1}{M}\left(E - R_1 i_1 - L_1 \frac{di_1}{dt}\right) \tag{6}$$

which, differentiated, becomes:

$$\frac{d^2 i_2}{dt^2} = \frac{1}{M}\left(-R_1 \frac{di_1}{dt} - L_1 \frac{d^2 i_1}{dt^2}\right) \tag{7}$$

Substituting (6) and (7) into equation (5) gives:

$$\frac{R_2}{M}\left(E - R_1 i_1 - L_1 \frac{di_1}{dt}\right) + \frac{L_2}{M}\left(-R_1 \frac{di_1}{dt} - L_1 \frac{d^2 i_1}{dt^2}\right) + M \frac{d^2 i_1}{dt^2} = 0$$

or:

$$R_2 E - R_1 R_2 i_1 - R_2 L_1 \frac{di_1}{dt} - R_1 L_2 \frac{di_1}{dt} - L_1 L_2 \frac{d^2 i_1}{dt^2} + M^2 \frac{d^2 i_1}{dt^2} = 0$$

which combined and rearranged yields:

$$\frac{d^2 i_1}{dt^2} + \left(\frac{R_1 L_2 + R_2 L_1}{L_1 L_2 - M^2}\right)\frac{di_1}{dt} + \frac{R_1 R_2}{(L_1 L_2 - M^2)} i_1 = \frac{R_2 E}{(L_1 L_2 - M^2)} \tag{8}$$

It will be noted that equation (8) is a typical differential equation of the second order having constant coefficients, and therefore admits of a solution for i_1 directly. Considering i_1 to be composed of its two components i_s and i_t:

$$i_1 = i_{1s} + i_{1t} \tag{9}$$

By inspection, the particular integral, i_{1s}, is seen to equal E/R_1. The complementary function i_{1t} may be found by methods which have previously been explained.

$$i_{1t} = A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \tag{10}$$

Substituting the value of i_{1t} in equation (8) yields:

$$\alpha_1,\ \alpha_2 = -\left[\frac{R_1L_2+R_2L_1}{2(L_1L_2-M^2)}\right] \pm \sqrt{\left[\frac{(R_1L_2+R_2L_1)}{2(L_1L_2-M^2)}\right]^2 - \frac{R_1R_2}{(L_1L_2-M^2)}} \qquad (11)\ (12)$$

It may be shown that the term under the radical sign will always be a positive quantity. Hence α_1 and α_2 will always take the form of real numbers, and the expression for current becomes:

$$i_1 = \frac{E}{R_1} + A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \tag{13}$$

Substituting one of the boundary conditions, namely, $i_1 = 0$ at $t = 0$, in (13) results in the following relation between A_1 and A_2:

$$A_1 + A_2 = -\frac{E}{R_1} \tag{14}$$

A determination of A_1 and A_2 requires that another relationship be established. An expression for i_2 in terms of A_1 and A_2 may be obtained. Substituting for $\frac{di_2}{dt}$ in equation (4), the following is obtained:

$$R_2i_2 + \frac{L_2}{M}\left(E - R_1i_1 - L_1\frac{di_1}{dt}\right) + M\frac{di_1}{dt} = 0$$

from which

$$i_2 = \frac{R_1L_2}{R_2M}i_1 + \left(\frac{L_1L_2}{R_2M} - \frac{M}{R_2}\right)\frac{di_1}{dt} - \frac{L_2E}{R_2M}$$

or

$$i_2 = \frac{R_1L_2}{R_2M}i_1 + \frac{(L_1L_2 - M^2)}{R_2M}\frac{di_1}{dt} - \frac{EL_2}{R_2M} \tag{15}$$

Substituting for i_1 and $\frac{di_1}{dt}$ in (15) and imposing the second boundary condition, $i_2 = 0$ at $t = 0$:

$$0 = R_1L_2\left[\frac{E}{R_1} + A_1 + A_2\right] + [L_1L_2 - M^2]\,[\alpha_1A_1 + \alpha_2A_2] - EL_2$$

or

$$0 = EL_2 + R_1L_2A_1 + R_1L_2A_2 + \alpha_1L_1L_2A_1 + \alpha_2L_1L_2A_2 - \alpha_1A_1M^2 - \alpha_2A_2M^2 - EL_2$$

which, simplified, reduces to:

$$A_1(R_1L_2 + \alpha_1L_1L_2 - \alpha_1M^2) + A_2(R_1L_2 + \alpha_2L_1L_2 - \alpha_2M^2) = 0 \quad (16)$$

From the simultaneous solution of equations (14) and 16):

$$A_1 = \frac{E}{\alpha_1 - \alpha_2}\left[\frac{L_2}{L_1L_2 - M^2} + \frac{\alpha_2}{R_1}\right] \quad (17)$$

and

$$A_2 = \frac{E}{\alpha_2 - \alpha_1}\left[\frac{L_2}{L_1L_2 - M^2} + \frac{\alpha_1}{R_1}\right] \quad (18)$$

Thus the expression for i_1, equation (13), is completely determined in terms of the applied voltage and the circuit parameters.

The expression for i_2 is at once obtainable:

$$i_2 = B_1\epsilon^{\alpha_1 t} + B_2\epsilon^{\alpha_2 t} \quad (19)$$

where

$$B_1 = -\frac{EM}{(\alpha_1 - \alpha_2)(L_1L_2 - M^2)} \quad (20)$$

$$B_2 = \frac{EM}{(\alpha_1 - \alpha_2)(L_1L_2 - M^2)} \quad (21)$$

i_2 reaches its negative maximum at the time $\dfrac{di_2}{dt} = 0$

$$\frac{di_2}{dt} = B_1\alpha_1\epsilon^{\alpha_1 t} + B_2\alpha_2\epsilon^{\alpha_2 t}$$

$$t_{(\max.\, i_2)} = \frac{1}{\alpha_1 - \alpha_2}\log\frac{\alpha_2}{\alpha_1} \quad (22)$$

With α_1, α_2, A_1, A_2, B_1, and B_2 properly evaluated, equations (13) and (19) are suitable solutions of i_1 and i_2 for any particular set of parameters; but in generalized form the expressions do not lend themselves readily to physical interpretation. Detailed study of the components will reveal the general nature of the current variations and also some of the effects of the magnetic coupling. For example, the current in circuit 2 is the sum of positive and negative exponential

components that are equal in magnitude at $t = 0$. The negative exponential term, $B_1\epsilon^{\alpha_1 t}$, is damped out at a slower rate than the positive term, $B_2\epsilon^{\alpha_2 t}$, because $\alpha_1 < \alpha_2$. The magnitudes of B_1 and B_2 are very closely related to the magnitude of M, and the damping factors, α_1 and α_2, are dependent to some extent upon the value of the magnetic coupling that exists between the two circuits.

It will further be observed that R_2 is a significant factor in determining the values of α_1 and α_2. Since α_1 and α_2 govern the rate of growth of i_1 as well as that of i_2, the nature of the i_1 variation may be controlled to a considerable extent by the magnitude of R_2. Advantage is taken of this fact in the construction of certain types of selective relays used in the communication field. The secondary circuit in such cases usually consists of a short-circuiting collar, the resistance of which helps to control the rate of growth of the primary current.

Numerical Example.—Let $R_1 = 1.7$ ohms, $L_1 = 0.017$ henry, $M = 0.012$ henry, $R_2 = 1.7$ ohms, $L_2 = 0.016$ henry, and $E = 4$ volts. With the above circuit parameters and applied voltage:

$$\alpha_1 = -60 \qquad \alpha_2 = -378$$
$$A_1 = -1.22 \qquad A_2 = -1.13$$
$$B_1 = -1.18 \qquad B_2 = 1.18$$

The expressions for i_1 and i_2, respectively, become:

$$i_1 = 2.35 - 1.22\epsilon^{-60t} - 1.13\epsilon^{-378t} \tag{23}$$

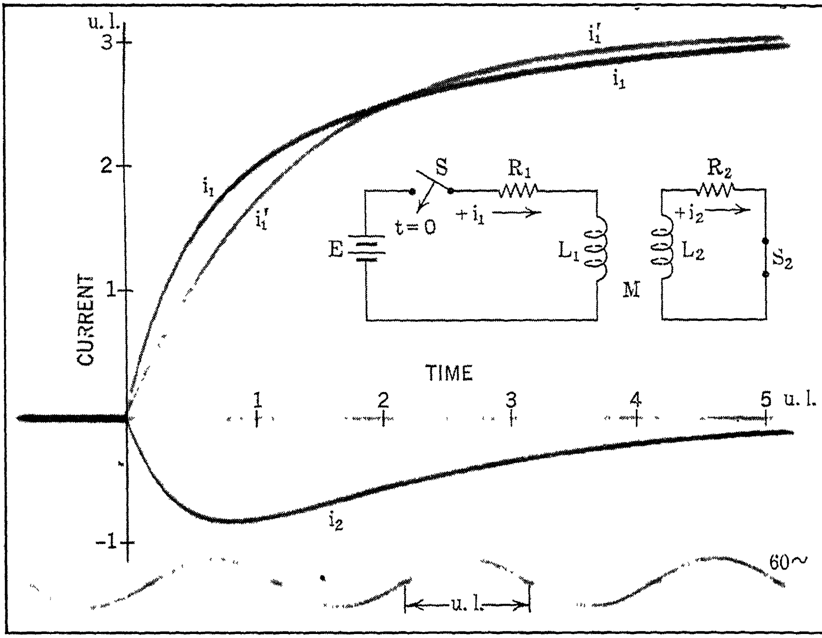
$$i_2 = -1.18\epsilon^{-60t} + 1.18\epsilon^{-378t} \tag{24}$$

With the secondary circuit open the expression for i_1 is

$$i'_1 = 2.35(1 - \epsilon^{-100t}) \tag{25}$$

A comparison of i_1 with i'_1 in plotted form will reveal the effect of the coupled secondary circuit upon the rate of growth of the primary current.

Oscillographic Verification.—The applied voltage and circuit parameters used in the above numerical example are approximately the same as those employed in connection with Oscillogram 1. Therefore, the graphs shown represent the plotted solutions of equations (23), (24), and (25) to a fair degree of accuracy. The i'_1 graph is the growth of current in circuit 1 when circuit 1 is not influenced by circuit 2. By means of double exposure the i'_1 variation has been superimposed on the photographic records of i_1 and i_2.



OSCILLOGRAM 1.

Primary current (i_1) and secondary current (i_2) in the $R_1L_1M\ R_2L_2$ combination. i_1' is the primary current when the secondary is open.

$E = 4$ volts. $R_1 = 1.7$ ohms. $L_1 = 0.017$ henry. $M = 0.012$ henry.

$R_2 = 1.7$ ohms. $L_2 = 0.016$ henry.

i_1 and i_1' calibration $= 0.75$ amp. per u. l.

i_2 calibration $= 0.81$ amp. per u. l.

INDUCTIVELY COUPLED PARALLEL BRANCHES

Physical Considerations.—According to the convention of signs previously adopted the counter-voltage of mutual induction is a positive quantity in the case of separate circuits. If, however, the branches between which mutual magnetic coupling exists are conductively connected the sign of the counter-voltages may be either plus or minus. The mode of winding and the space position of the branches with respect to one another will determine the sign of the counter-voltages of mutual induction. A brief study of the two RL branches in parallel will illustrate the effect that the sign of M may have upon the transient behavior of circuits.

The two parallel branches shown in Fig. 2 are magnetically coupled, and the position of the coils with respect to one another, together with the style of winding, is assumed to be as indicated. The sign of i_2 is

defined by the polarity of the driving voltage E. It will be noted that $+ i_2$ establishes flux in the L_1 coil in a direction opposite to that produced in the L_1 coil by $+ i_1$. Therefore, the counter-voltage, $M \frac{di_2}{dt}$, acts in an opposite direction to the counter-voltage of self-induction, $L_1 \frac{di_1}{dt}$, and the same is true of $M \frac{di_1}{dt}$ and $L_2 \frac{di_2}{dt}$. In accordance with convention, the sign of M is minus. A reversal of the terminal connections of the L_2 coil will result in a reversal of the sign of M.

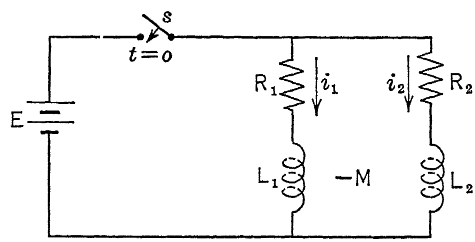


FIG. 2.—Inductively coupled parallel branches.

The general effects of the two connections may be determined by direct application of Lenz's law. With the coils connected as shown in Fig. 2 the effect of the self-inductance is diminished by the mutual inductance. i_1 and i_2 are thereby accelerated in their building-up process and may in fact overreach their respective E/R values. With the opposite terminal connections, $(+ M)$, the inductive effects of the branches are increased by the magnetic coupling. $M \frac{di_2}{dt}$ acts in the same direction in branch 1 as $L_1 \frac{di_1}{dt}$, which is of course, opposed to the driving voltage E. Under certain conditions $M \frac{di_2}{dt}$ may even establish a negative current in branch 1 during the early part of the transient period. With proper adjustment of the parameters wide differences in the current-time graphs may therefore be obtained.

Mathematical Analysis. *The Minus M Connection.*—With the connection such that M is minus:

$$L_1 \frac{di_1}{dt} + R_1 i_1 - M \frac{di_2}{dt} = E \tag{26}$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 - M \frac{di_1}{dt} = E \tag{27}$$

Operational methods will be employed in finding the explicit expressions for i_1 and i_2. In operational form equations (26) and (27) become:

$$L_1 p i_1 + R_1 i_1 - M p i_2 = E \tag{28}$$

$$L_2 p i_2 + R_2 i_2 - M p i_1 = E \tag{29}$$

From equation (28):

$$i_2 = \frac{L_1 p i_1 + R_1 i_1 - E}{Mp} \tag{30}$$

Substituting in equation (29) and rearranging:

$$i_1 = \frac{E(L_2 p + Mp + R_2)}{(L_1 L_2 - M^2)p^2 + (R_1 L_2 + R_2 L_1)p + R_1 R_2} \tag{31}$$

From equation (28):

$$i_1 = \frac{E + Mpi_2}{L_1 p + R_1} \tag{32}$$

From equation (29):

$$i_2 = \frac{E + Mpi_1}{L_2 p + R_2} \tag{33}$$

Substituting (32) in equation (33) and rearranging:

$$i_2 = \frac{E(L_1 p + Mp + R_1)}{(L_1 L_2 - M^2)p^2 + (R_1 L_2 + R_2 L_1)p + R_1 R_2} \tag{34}$$

By the expansion theorem:

$$i_1 = \frac{E}{R_1} + A_1 \epsilon^{\alpha_1 t} + A_2 \epsilon^{\alpha_2 t} \tag{35}$$

$$i_2 = \frac{E}{R_2} + B_1 \epsilon^{\alpha_1 t} + B_2 \epsilon^{\alpha_2 t} \tag{36}$$

where

$$\alpha_1, \alpha_2 = \frac{-(R_1 L_2 + R_2 L_1)}{2(L_1 L_2 - M^2)} \pm \sqrt{\frac{(R_1 L_2 + R_2 L_1)^2}{4(L_1 L_2 - M^2)^2} - \frac{R_1 R_2}{(L_1 L_2 - M^2)}} \tag{37, 38}$$

$$A_1 = \frac{E}{\alpha_1 - \alpha_2}\left[\frac{L_2 + M}{L_1 L_2 - M^2} + \frac{\alpha_2}{R_1}\right] \tag{39}$$

$$A_2 = \frac{E}{\alpha_2 - \alpha_1}\left[\frac{L_2 + M}{L_1 L_2 - M^2} + \frac{\alpha_1}{R_1}\right] \tag{40}$$

$$B_1 = \frac{E}{\alpha_1 - \alpha_2}\left[\frac{L_1 + M}{L_1 L_2 - M^2} + \frac{\alpha_2}{R_2}\right] \tag{41}$$

$$B_2 = \frac{E}{\alpha_2 - \alpha_1}\left[\frac{L_1 + M}{L_1 L_2 - M^2} + \frac{\alpha_1}{R_2}\right] \tag{42}$$

Equations (35) and (36) are much better adapted to numerical solutions in particular cases than to general interpretation.

The Plus M Connection.—When M is positive:

$$L_1pi_1 + R_1i_1 + Mpi_2 = E \tag{43}$$

$$L_2pi_2 + R_2i_2 + Mpi_1 = E \tag{44}$$

and

$$i_1 = \frac{E(L_2p - Mp + R_2)}{(L_1L_2 - M^2)p^2 + (R_1L_2 + R_2L_1)p + R_1R_2} \tag{45}$$

$$i_2 = \frac{E(L_1p - Mp + R_1)}{(L_1L_2 - M^2)p^2 + (R_1L_2 + R_2L_1)p + R_1R_2} \tag{46}$$

The solutions of i_1 and i_2 take the same general form as given for the minus M connection, thus:

$$i_1 = \frac{E}{R_1} + D_1\epsilon^{\alpha_1 t} + D_2\epsilon^{\alpha_2 t} \tag{47}$$

$$i_2 = \frac{E}{R_2} + F_1\epsilon^{\alpha_1 t} + F_2\epsilon^{\alpha_2 t} \tag{48}$$

where

$$D_1 = \frac{E}{\alpha_1 - \alpha_2}\left[\frac{L_2 - M}{L_1L_2 - M^2} + \frac{\alpha_2}{R_1}\right] \tag{49}$$

$$D_2 = \frac{E}{\alpha_2 - \alpha_1}\left[\frac{L_2 - M}{L_1L_2 - M^2} + \frac{\alpha_1}{R_1}\right] \tag{50}$$

$$F_1 = \frac{E}{\alpha_1 - \alpha_2}\left[\frac{L_1 - M}{L_1L_2 - M^2} + \frac{\alpha_2}{R_2}\right] \tag{51}$$

$$F_2 = \frac{E}{\alpha_2 - \alpha_1}\left[\frac{L_1 - M}{L_1L_2 - M^2} + \frac{\alpha_1}{R_2}\right] \tag{52}$$

α_1 and α_2 are not changed by the reversal of the sign of M. Their values are given in terms of the circuit parameters in equations (37) and (38).

The A's, B's, D's, and F's are the $\left[\dfrac{Y(p)}{p\dfrac{dZ(p)}{dp}}\right]$ constants of Heaviside's expansion theorem and are equivalent to the constants of integration that appear when the conventional method of solution is employed. The current-time graphs are largely dependent upon the relative magnitude and algebraic sign of these coefficients. Comparison of equations (31) and (45) will show the manner in which the reversed sign of M may greatly influence the values of the A's as compared with the values

of the D's. If M is greater than L_2, and R_2 is not a significant factor, A_1 will be opposite in sign and, in general, different in magnitude from D_1. Similar differences will exist between A_2 and D_2. Thus the transient variation of i_1 when M is positive may be expected to differ considerably from the variation when M is negative.

Numerical Example.—A numerical example in which M is greater than L_2 will serve to illustrate how the relative magnitudes and algebraic signs of the factors are changed by the reversed sign of M.

Let:

$$\begin{aligned} R_1 &= 3.86 \text{ ohms} \\ L_1 &= 0.093 \text{ henry} \\ R_2 &= 0.965 \text{ ohm} \\ L_2 &= 0.011 \text{ henry} \\ M &= 0.026 \text{ henry} \end{aligned}$$

From the equations previously derived, the following numerical magnitudes may be obtained:

$$\begin{aligned} \alpha_1 &= -30.6 & \alpha_2 &= -351.6 \\ A_1 &= 0.049E & A_2 &= -0.308E \\ B_1 &= -0.064E & B_2 &= -0.973E \\ D_1 &= -0.419E & D_2 &= 0.159E \\ F_1 &= -0.533E & F_2 &= 0.505E \end{aligned}$$

For minus M:

$$i_1 = E(0.259 + 0.049\epsilon^{-30.6t} - 0.308\epsilon^{-351.6t}) \tag{53}$$

$$i_2 = E(1.036 - 0.064\epsilon^{-30.6t} - 0.973\epsilon^{-351.6t}) \tag{54}$$

For positive M:

$$i_1 = E(0.259 - 0.419\epsilon^{-30.6t} + 0.159\epsilon^{-351.6t}) \tag{55}$$

$$i_2 = E(1.036 - 0.533\epsilon^{-30.6t} - 0.505\epsilon^{-351.6t}) \tag{56}$$

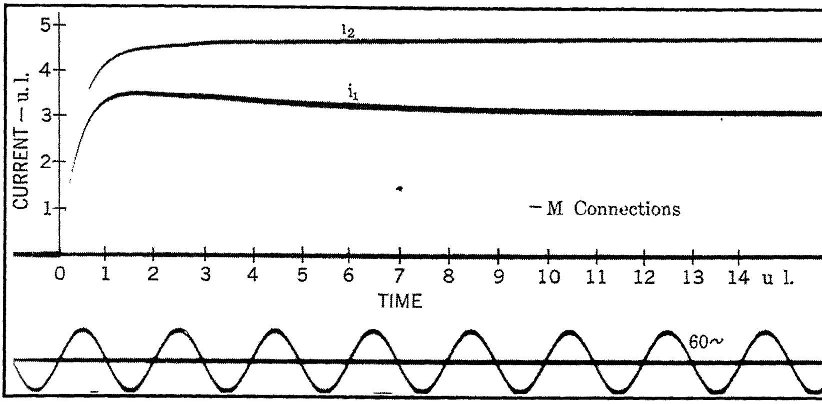
Plotted solutions of the above equations are shown in Fig. 3 for the case of unit voltage applied. The difference between the current-time graphs of i_1 for the two signs of M is extremely marked. In the case of positive M, i_1 is negative during the early stages, reaching a minimum value at the time $\left(\frac{di_1}{dt}\right) = 0.$

$$t \text{ (minimum } i_1) = 0.0046 \text{ second.}$$

In the case of minus M, i_1 reaches positive values that are greater than E/R_1. The time at which maximum i_1 occurs is 0.0133 second.

Oscillographic Verification.—Using the same circuit parameters as in the above illustrative example, the following two oscillograms verify the physical and mathematical deductions.

Oscillogram 2 illustrates the experimentally determined current-time graphs of i_1 and i_2 for the $-M$ connection of the parallel branches.



OSCILLOGRAM 2.

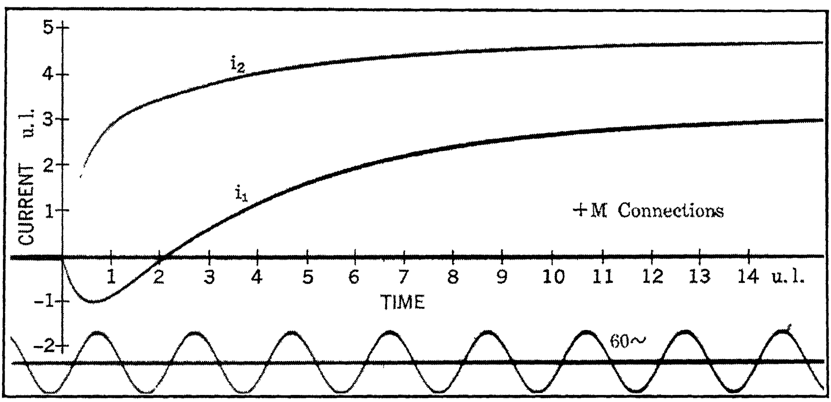
Inductively Coupled Parallel Branches. See Fig. 2.

$E = 6.0$ volts. $R_1 = 3.86$ ohms. $L_1 = 0.093$ henry.

$R_2 = 0.96$ ohm. $L_2 = 0.011$ henry. $M = -0.026$ henry.

i_1 calibration $= 0.53$ amp. per u. l.

i_2 calibration $= 1.37$ amp. per u. l.



OSCILLOGRAM 3.

Inductively Coupled Parallel Branches. The circuit parameters are the same as in Oscillogram 2 except for the sign of M.

The corresponding graphs with the $+M$ connection of the branches are shown in Oscillogram 3. In order to show better the marked differences between the two i_1 graphs, the i_1 galvanometer was made more sensitive than the i_2 galvanometer. If a comparison between the oscillograms and the numerically determined solutions shown in Fig. 3 is made, the i_1 and i_2 calibrations should therefore be noted.

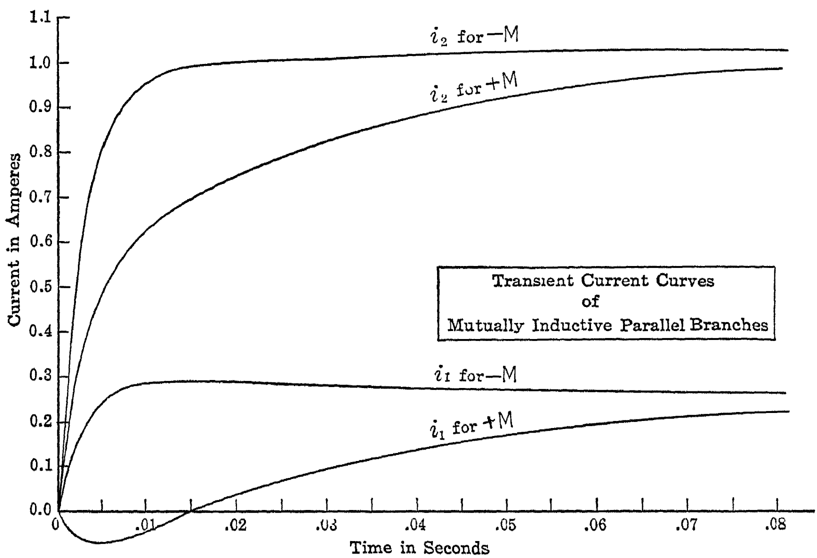
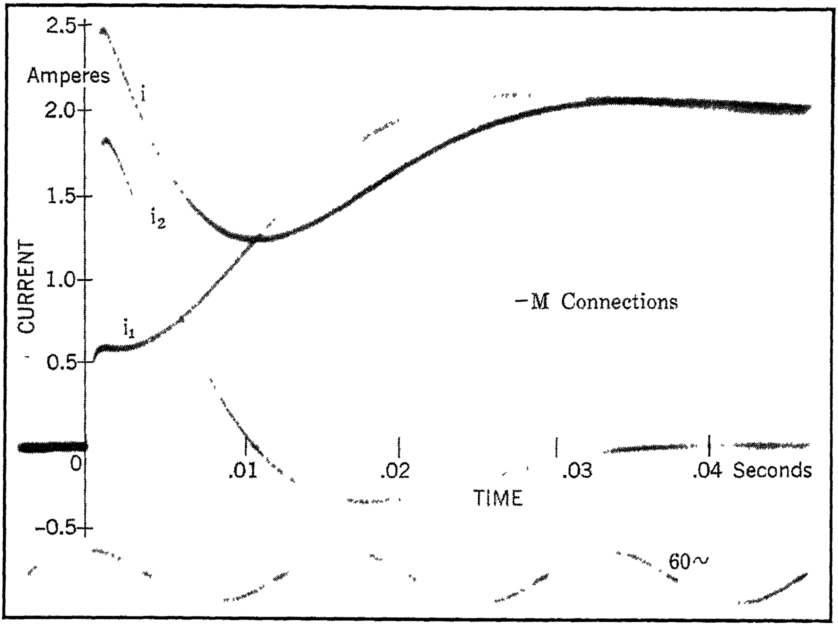


FIG. 3.—Plotted solutions of equations (53), (54), (55), and (56).

MUTUAL INDUCTANCE IN SERIES-PARALLEL CIRCUITS

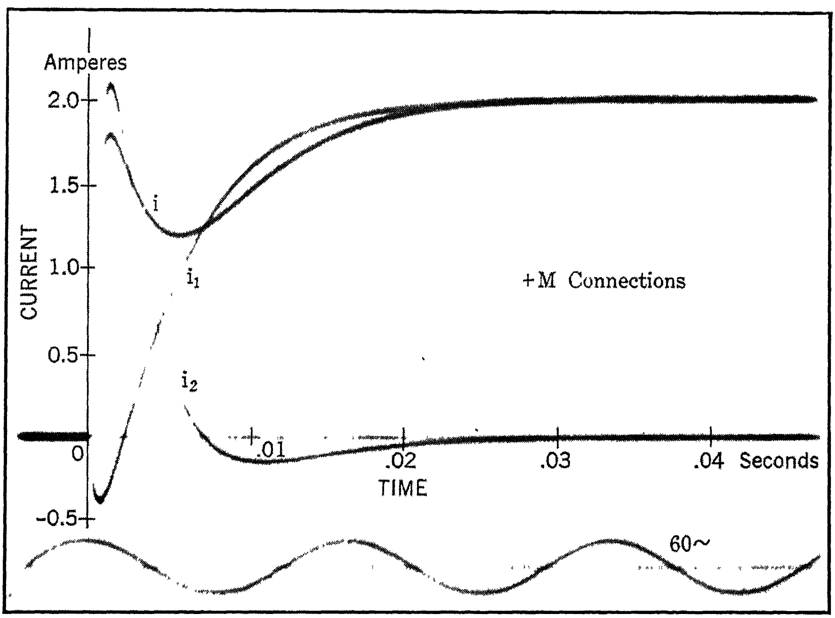
The transient effects produced by magnetic coupling between the branches of a series-parallel circuit are extremely varied. Fig. 4 is the circuit diagram of a simple series-parallel combination wherein an RL branch is inductively coupled to an RLC branch.

Oscillograms 4 and 5 illustrate the varied nature of the current-time graphs as well as the striking effect that the sign of M plays in determining the natural response of electrical circuits. Oscillogram 4 shows the variations of i_1, i_2, and i for a particular set of circuit parameters when the connections are such that M is negative. Oscillogram 5 illustrates the same variations with the terminal connections of the L coils such that M is positive. The principles pertaining to mutual induction that have previously been discussed are especially well illustrated.



OSCILLOGRAM 4.

Illustrating the marked effects of $-M\frac{di_2}{dt}$ upon the growth of i_1. Circuit diagram and parameters are given in connection with Fig. 4.



OSCILLOGRAM 5.

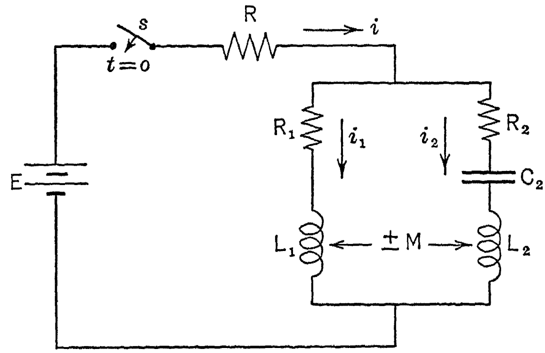


FIG. 4.—Series-parallel circuit wherein M is an appreciable factor.
E = 30.5 volts. R = 9.1 ohms. R_1 = 6.75 ohms. L_1 = 0.093 henry.
R_2 = 2.5 ohms. L_2 = 0.011 henry. C_2 = 390μf. M = 0.026 henry.

THE $R_1L_1C_1M$ R_2L_2 COMBINATION

Close magnetic coupling greatly alters the natural response of an otherwise oscillatory series circuit. Among the major effects of the coupling are: (1) the shortening of the period of oscillation, and (2) the increased rate at which the oscillation is damped. Other effects are the

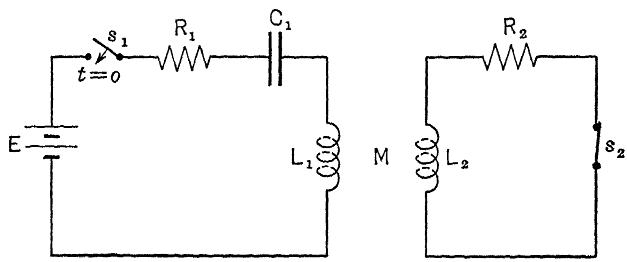


FIG. 5.—The $R_1L_1C_1M$ R_2L_2 combination.

increased maximum current values during the early stages and the unsymmetrical type of oscillation that follows.

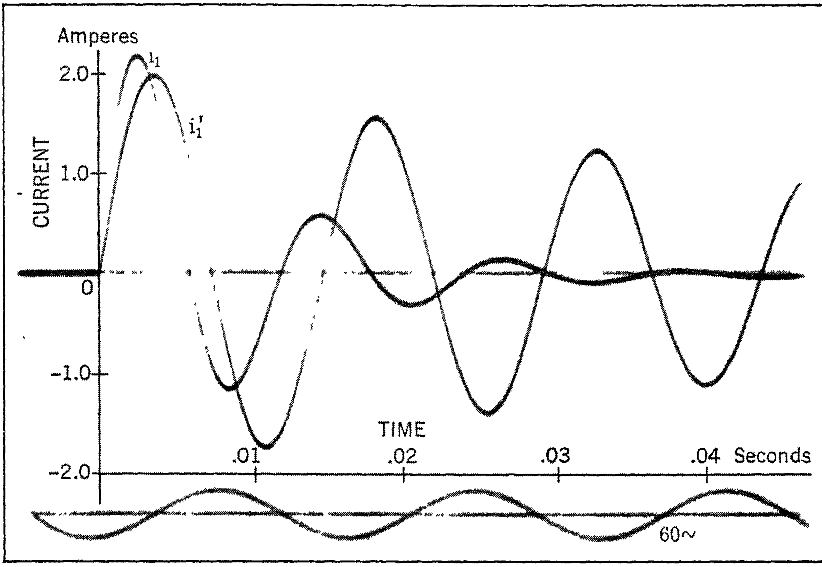
The current-time graphs of i_1 and i_2 for the combination shown in Fig. 5 are not symmetrical about the zero line. Assuming that the parameters are such as to permit energy interchanges, the graphs consist of a damped oscillatory variation superimposed on a subsiding exponential curve. Mathematical analysis will show that:

$$i_1 = A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} + A_3\epsilon^{\alpha_3 t} \tag{57}$$

and that:

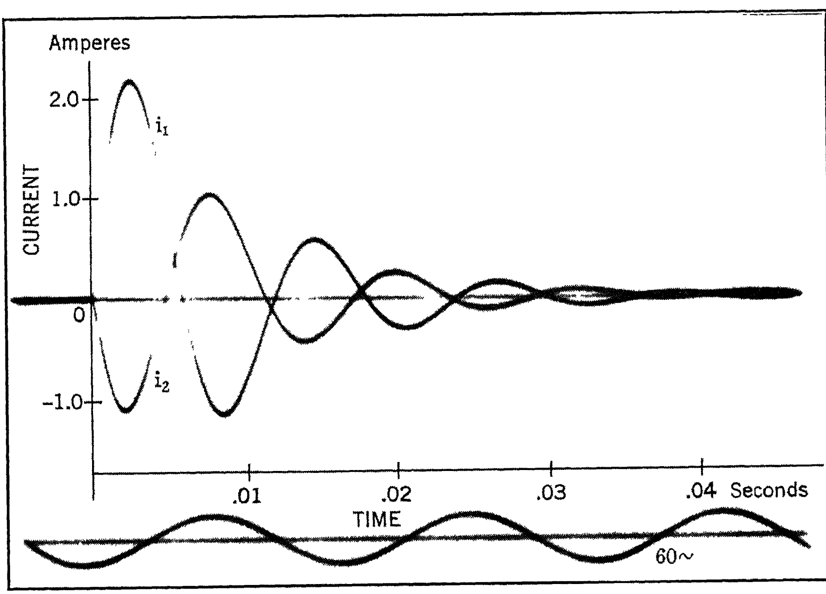
$$i_2 = B_1\epsilon^{\alpha_1 t} + B_2\epsilon^{\alpha_2 t} + B_3\epsilon^{\alpha_3 t} \tag{58}$$

Two of the exponential terms in each of the above expressions combine



OSCILLOGRAM 6.

Illustrating the effect of magnetic coupling upon an oscillatory circuit. (See Fig. 5.)
i_1' is the primary current when circuit 2 is open.
i_1 is the primary current when circuit 2 is closed.
$E = 18$ volts. $R_1 = 0.5$ ohm. $L_1 = 0.0172$ henry. $C_1 = 317\mu$f.
$R_2 = 3.8$ ohms. $L_2 = 0.0159$ henry. $M = 0.0115$ henry.

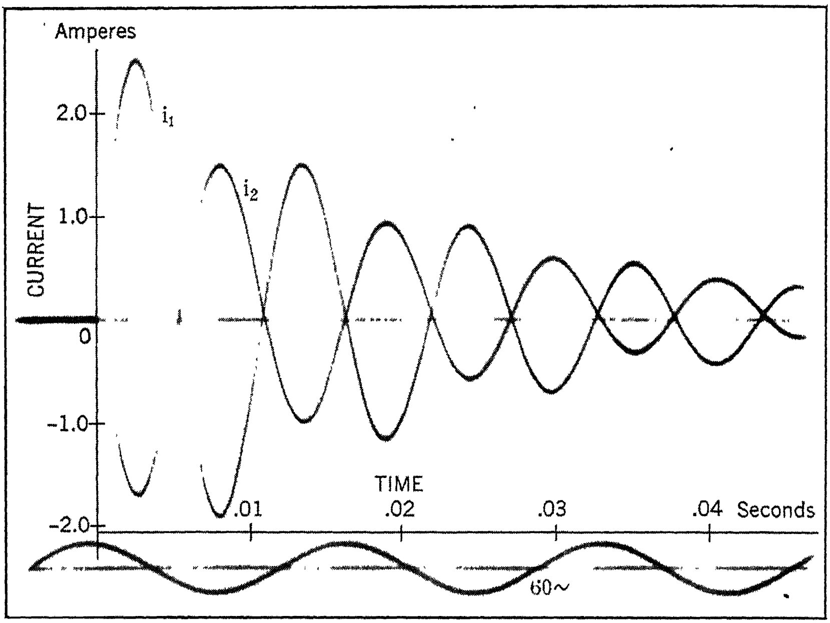


OSCILLOGRAM 7.

Simultaneous variations of i_1 and i_2 in the $R_1L_1C_1M\ R_2L_2$ combination.
Same parameters as in Oscillogram 6.

to form an oscillatory component while the remaining one persists merely as a $K\epsilon^{-at}$ term. The details connected with the actual evaluation are reserved for student analysis.

Oscillographic Demonstration.—The manner in which mutual inductance affects the primary current of Fig. 5 is illustrated by Oscillogram 6. i_1' is the natural response of the RLC primary when circuit 2 is open. i_1 is the graph of the primary current when circuit 2 is closed



Oscillogram 8.

Simultaneous variations of i_1 and i_2 in the $R_1L_1C_1M\ R_2L_2$ combination for a reduced value of R_2. Same parameters as in Oscillograms 6 and 7 except that R_2 is here reduced to 0.35 ohm.

and tightly coupled with circuit 1. The change in the frequency of oscillation as well as the change in the rate of damping is distinctly in evidence.

Simultaneous graphs of i_1 and i_2 for the same parameters as employed above are shown in Oscillogram 7. The fact that the current variations consist of a damped oscillatory component plus a decaying exponential term is indicated by the zero crossings of the graphs. The resistance of the secondary (circuit 2) is largely responsible for the unsymmetrical zero crossings.

Oscillogram 8 illustrates the effect of reducing R_2. Except for the

change in R_2 the circuit parameters are the same as in Oscillograms 6 and 7. The reduction in R_2 has greatly increased the symmetry of the current-time graphs with respect to the zero line. Comparison of Oscillograms 7 and 8 also reveals the extent to which R_2 governs the rate of subsidence. The latter oscillogram is an example of the precision with which the principles stated in Lenz's law operate.

THE $R_1L_1C_1M$ $R_2L_2C_2$ COMBINATION

Magnetically coupled RLC circuits have occupied an important place in the field of electrical engineering since the advent of wireless telegraphy; from the early spark transmitters and power buzzers down to the modern vacuum-tube oscillators, coupled circuits have played no small part in the development of wireless communication.

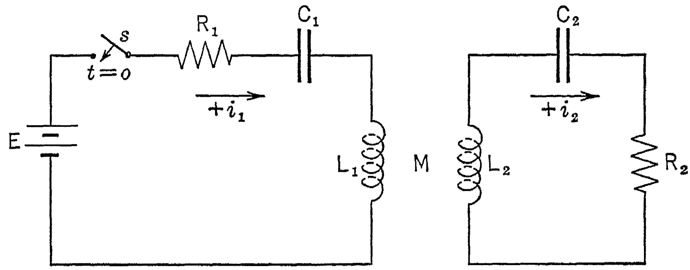


FIG. 6.—The $R_1L_1C_1M$ $R_2L_2C_2$ combination.

Mathematical Analyses.—Referring to Fig. 6, the expressions for dynamic equilibrium in circuits 1 and 2 are:

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{q_1}{C_1} + M \frac{di_2}{dt} = E \tag{59}$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{q_2}{C_2} + M \frac{di_1}{dt} = 0 \tag{60}$$

One method of solution of the above equations is that of successive differentiation which, if employed, leads to:

$$\frac{d^4 i_1}{dt^4} + K_1 \frac{d^3 i_1}{dt^3} + K_2 \frac{d^2 i_1}{dt^2} + K_3 \frac{di_1}{dt} + K_4 i_1 = 0 \tag{61}$$

and

$$\frac{d^4 i_2}{dt^4} + K_1 \frac{d^3 i_2}{dt^3} + K_2 \frac{d^2 i_2}{dt^2} + K_3 \frac{di_2}{dt} + K_4 i_2 = 0 \tag{62}$$

where

$$K_1 = \frac{(R_1L_2 + R_2L_1)}{(L_1L_2 - M^2)} \tag{63}$$

$$K_2 = \frac{\left(\frac{L_1}{C_2} + \frac{L_2}{C_1} + R_1R_2\right)}{(L_1L_2 - M^2)} \tag{64}$$

$$K_3 = \frac{\left(\frac{R_1}{C_2} + \frac{R_2}{C_1}\right)}{(L_1L_2 - M^2)} \tag{65}$$

$$K_4 = \frac{1}{(L_1L_2 - M^2)C_1C_2} \tag{66}$$

From the elementary theory of linear differential equations:

$$i_1 = A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} + A_3\epsilon^{\alpha_3 t} + A_4\epsilon^{\alpha_4 t} \tag{67}$$

$$i_2 = B_1\epsilon^{\alpha_1 t} + B_2\epsilon^{\alpha_2 t} + B_3\epsilon^{\alpha_3 t} + B_4\epsilon^{\alpha_4 t} \tag{68}$$

In any particular case the α's may be expressed in terms of the circuit parameters. The chief difficulty is encountered in the evaluation of the eight constants of integration, and until at least the general form of the constants has been determined, equations (67) and (68) are meaningless from a physical point of view.

The Operational Solution.—Inasmuch as operational expressions are employed in the numerical examples that follow, a brief outline of the method is given below. In terms of the Heaviside operator, equations (59) and (60) become:

$$L_1pi_1 + R_1i_1 + \frac{i_1}{pC_1} + Mpi_2 = E \tag{69}$$

$$L_2pi_2 + R_2i_2 + \frac{i_2}{pC_2} + Mpi_1 = 0 \tag{70}$$

from which:

$$i_2 = \frac{-Mpi_1}{\left(L_2p + R_2 + \frac{1}{pC_2}\right)} \tag{71}$$

$$i_1\left(L_1p + R_1 + \frac{1}{pC_1}\right) - \frac{M^2p^2i_1}{\left(L_2p + R_2 + \frac{1}{pC_2}\right)} = E \tag{72}$$

and

$$i_1 = \frac{E\left(L_2p + R_2 + \dfrac{1}{pC_2}\right)}{\left\{\begin{array}{c}(L_1L_2 - M^2)p^2 + (L_1R_2 + L_2R_1)p + \left(\dfrac{L_1}{C_2} + \dfrac{L_2}{C_1} + R_1R_2\right) \\ + \left(\dfrac{R_1}{C_2} + \dfrac{R_2}{C_1}\right)\dfrac{1}{p} + \dfrac{1}{C_1C_2p^2}\end{array}\right\}} \tag{73}$$

$$= \frac{E\left(L_2p^3 + R_2p^2 + \dfrac{p}{C_2}\right)\Big/(L_1L_2 - M^2)}{p^4 + K_1p^3 + K_2p^2 + K_3p + K_4} \tag{74}$$

and

$$i_2 = \frac{-EMp^3/(L_1L_2 - M^2)}{p^4 + K_1p^3 + K_2p^2 + K_3p + K_4} \tag{75}$$

The values of K_1, K_2, K_3, and K_4 are given in (63), (64), (65), and (66) in terms of the circuit parameters.

Evaluation of i_1.—Applying the expansion theorem:

$$i_1 = E\left[\frac{Y(0)}{Z(0)} + \sum_{\substack{p = p_1 \\ = p_2 \\ = p_3 \\ = p_4}} \frac{Y(p)\,\epsilon^{pt}}{p\dfrac{dZ(p)}{dp}}\right] \tag{76}$$

where

$$Y(p) = \frac{L_2p^3 + R_2p^2 + \dfrac{p}{C_2}}{(L_1L_2 - M^2)} \tag{77}$$

$$Z(p) = p^4 + K_1p^3 + K_2p^2 + K_3p + K_4 \tag{78}$$

p_1, p_2, p_3, and p_4 are the roots of $Z(p) = 0$.

It will be observed that:

$$\frac{Y(0)}{Z(0)} = 0 \tag{79}$$

This is compatible with physical facts since the steady-state value of i_1 is obviously equal to zero.

Equation (67) thus becomes:

$$i_1 = A_1\epsilon^{p_1} + A_2\epsilon^{p_2t} + A_3\epsilon^{p_3t} + A_4\epsilon^{p_4t} \tag{80}$$

where

$$A_1 = E\left[\frac{Y(p)}{p\dfrac{dZ(p)}{dp}}\right]_{\text{for } p = p_1}$$

$$= E\left[\frac{\left(L_2p^2 + R_2p + \dfrac{1}{C_2}\right)\Big/(L_1L_2 - M^2)}{4p^3 + 3K_1p^2 + 2K_2p + K_3}\right]_{\text{for } p = p_1} \tag{81}$$

Corresponding expressions for A_2, A_3, and A_4 may be written Assuming that, in any particular case, p_1, p_2, p_3, and p_4 have been determined, the evaluation of the constants of integration is a straightforward procedure.

In general, if the circuit parameters are such that i_1 is oscillatory, p_1, p_2, p_3, and p_4 consist of two sets of conjugate roots.

Thus

$$p_1 = -a_1 + jb_1 \tag{82}$$

$$p_2 = -a_1 - jb_1 \tag{83}$$

$$p_3 = -a_2 + jb_2 \tag{84}$$

$$p_4 = -a_2 - jb_2 \tag{85}$$

If the roots of $Z(p) = 0$ are of the form given above, A_1 and A_2 are conjugates and likewise A_3 and A_4 are conjugates. That such must be the case is quite obvious. Otherwise equation (80) would, in general, possess imaginary terms—an incompatible situation in a physically realizable circuit. A neat mathematical proof of the conjugate nature of A_1 and A_2 and of A_3 and A_4 requires vector notation and will not be given at this point.

$$A_1 = A\epsilon^{j\sigma_1} \quad \text{and} \quad A_2 = A\epsilon^{-j\sigma_1}$$

$$A_3 = B\epsilon^{j\sigma_2} \quad \text{and} \quad A_4 = B\epsilon^{-j\sigma_2}$$

$$\begin{aligned} i_1 &= A\epsilon^{-a_1t}[\epsilon^{j(b_1t+\sigma_1)} + \epsilon^{-j(b_1t+\sigma_1)}] \\ &\quad + B\epsilon^{-a_2t}[\epsilon^{j(b_2t+\sigma_2)} + \epsilon^{-j(b_2t+\sigma_2)}] \end{aligned} \tag{86}$$

$$i_1 = 2A\epsilon^{-a_1t}\cos(b_1t + \sigma_1) + 2B\epsilon^{-a_2t}\cos(b_2t + \sigma_2) \tag{87}$$

It will be observed that i_1 is composed of two distinct oscillatory components. b_1 and b_2 are the natural angular velocities of the oscillations and a_1 and a_2 are their respective damping constants. The natural frequencies are:

$$f_1 = \frac{b_1}{2\pi} \quad \text{and} \quad f_2 = \frac{b_2}{2\pi}$$

It is by virtue of the two natural frequencies that the combination is inherently a " double band " pass filter.

Evaluation of i_2.—Applying the expansion theorem to equation (75):

$$Y(p) = \frac{-Mp^3}{(L_1L_2 - M^2)} \tag{88}$$

$$Z(p) = p^4 + K_1p^3 + K_2p^2 + K_3p + K_4 \tag{89}$$

p_1, p_2, p_3, and p_4 are the same as in the i_1 evaluation since $Z(p)$ is the same for both i_1 and i_2.

$$\frac{Y(0)}{Z(0)} = 0$$

Therefore

$$i_2 = D_1\epsilon^{p_1t} + D_2\epsilon^{p_2t} + D_3\epsilon^{p_3t} + D_4\epsilon^{p_4t} \tag{90}$$

$$D_1 = E\left[\frac{Y(p)}{p\frac{dZ(p)}{dp}}\right]_{\text{for } p=p_1}$$

$$= -E\left[\frac{Mp^2/(L_1L_2-M^2)}{4p^3+3K_1p^2+2K_2p+K_3}\right]_{\text{for } p=p_1} \tag{91}$$

Similar expressions may be written for D_2, D_3, and D_4.

$$D_1 = D\epsilon^{j\delta_1} \quad \text{and} \quad D_2 = D\epsilon^{-j\delta_1}$$

$$D_3 = F\epsilon^{j\delta_2} \quad \text{and} \quad D_3 = F\epsilon^{-j\delta_2}$$

The general expression for i_2 is similar in nature to the expression for i_1.

$$i_2 = 2D\epsilon^{-a_1t}\cos(b_1t+\delta_1) + 2F\epsilon^{-a_2t}\cos(b_2t+\delta_2) \tag{92}$$

The damping factors and natural angular velocities of i_1 and i_2 are equal, but the numerical coefficients that govern the magnitude of oscillation are distinctly different. Furthermore, the constant phase angles, δ_1 and δ_2, are not the same as σ_1 and σ_2 for i_1.

Equations (87) and (92), in addition to describing the general mode of variation of i_1 and i_2, greatly simplify numerical solutions in particular cases. And it is by way of numerical and oscillographic solutions that the $R_1L_1C_1M$ $R_2L_2C_2$ response is to be further analyzed.

Numerical Solution.—The parameters are so chosen that the uncoupled natural periods of circuit 1 and circuit 2 differ widely Let

$$R_1 = 3.5 \text{ ohms}, \quad L_1 = 0.093 \text{ henry}, \quad C_1 = 150\mu f$$

$$R_2 = 0.8 \text{ ohm}, \quad L_2 = 0.011 \text{ henry}, \quad C_2 = 168\mu f$$

$$\text{Tight coupling: } M = 0.026 \text{ henry}, \quad E = 30 \text{ volts.}$$

The uncoupled natural frequency and damping factor of circuit 1 are:

$$f_1' = 42.5 \text{ cycles per second}$$

$$\text{D.F.}_1' = \epsilon^{-18.8t}$$

Corresponding characteristics of circuit 2 are:

$$f_2' = 117 \text{ cycles per second}$$

$$\text{D.F.}_2' = \epsilon^{-36.4t}$$

The first step in the coupled circuit solution is the evaluation of the determinantal equation, $Z(p) = 0$. Substituting the values of the circuit parameters in equations (63), (64), (65), and (66),

$$p^4 + 326p^3 + 1.81\times10^6p^2 + 7.52\times10^7p + 1.142\times10^{11} = 0 \tag{93}$$

The roots of the above equations are: [1]

[1] The details connected with the solution are given in the Appendix, at which point the general theory of solution of fourth degree equations is considered.

$$p_1 = -146.9 + j1309 \tag{95}$$

$$p_2 = -146.9 - j1309 \tag{96}$$

$$p_3 = -16.1 + j256 \tag{97}$$

$$p_4 = -16.1 - j256 \tag{98}$$

Equations (87) and (92) show that p_1, p_2, p_3, and p_4 define the modes of variation of i_1 and i_2. The magnitude of the oscillations as well as the constant phase angles of the above equations depend upon the $\left[\dfrac{Y(p)}{p\dfrac{dZ(p)}{dp}}\right]$ terms of the operational solution.

Performing the operations indicated by equations (81) and (91) for each of the four roots yields:

$$A_1 = 0.259\epsilon^{-j87°53'} \qquad D_1 = 0.888\epsilon^{j90°58'}$$

$$A_2 = 0.259\epsilon^{j87°53'} \qquad D_2 = 0.888\epsilon^{-j90°58'}$$

$$A_3 = -0.528\epsilon^{j88°57'} \qquad D_3 = -0.173\epsilon^{j94°54'}$$

$$A_4 = -0.528\epsilon^{-j88°57'} \qquad D_4 = -0.173\epsilon^{-j94°54'}$$

The complete numerical expressions for i_1 and i_2 may now be written as follows:

$$\begin{aligned} i_1 &= 0.518\epsilon^{-146.9t}\cos(1309t - 87°\,53') \\ &\quad - 1.056\epsilon^{-16.1t}\cos(256t + 88°\,57') \end{aligned} \tag{99}$$

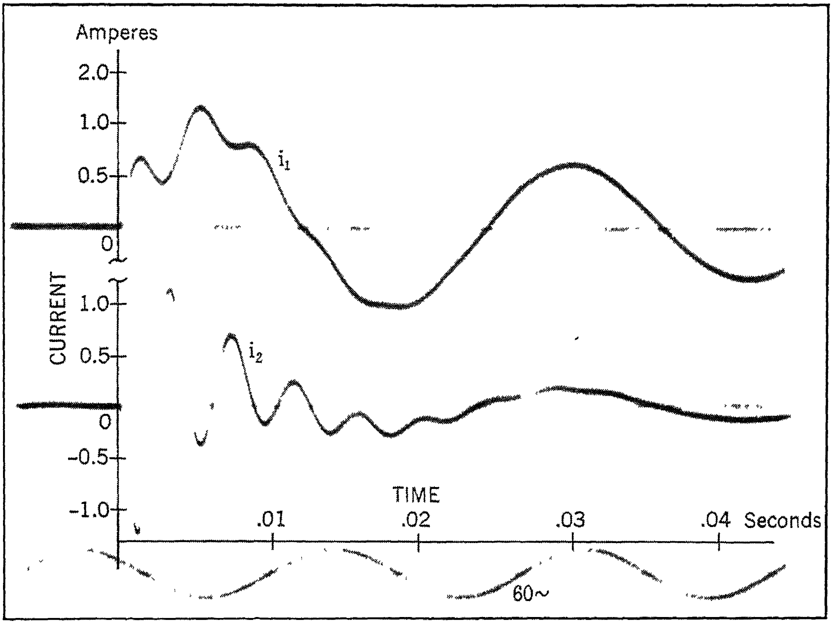
and

$$\begin{aligned} i_2 &= 1.776\epsilon^{-146.9t}\cos(1309t + 90°\,58') \\ &\quad - 0.346\epsilon^{-16.1t}\cos(256t + 94°\,54') \end{aligned} \tag{100}$$

Several effects of the magnetic coupling are evident. The two natural frequencies are 40.8 and 208 cycles per second. These are considerably different from the individual natural frequencies previously determined for circuit 1 and circuit 2. It will further be observed that the damping factors are affected as a result of the coupling. The higher-frequency component is much more predominant in i_2 than in i_1, and, during the early transient period, it is much greater than the lower-frequency component in the i_2 variation. Since, however, the higher-frequency component is damped so much more rapidly, the i_2 variation in the later transient period consists principally of the lower-frequency component.

Oscillographic Verification.—The primary and secondary currents in the $R_1L_1C_1M\ R_2L_2C_2$ combination having parameters similar to those of the preceding numerical example are shown in Oscillogram 9. Because of the similarity in parameters, the i_1 graph may be treated as the plotted solution of equation (99), and the i_2 graph as the plotted solution of equation (100). The main features of the numerical solution are illustrated. It will be observed that both graphs have component variations of two distinct frequencies. The lower-frequency component is much more pronounced in the primary circuit than the higher-

frequency component although the higher-frequency reflection can easily be seen during the early part of the transient period. The relatively large magnitude of the high-frequency component in the secondary current and the rapid rate at which it is damped are illustrated by the oscillographic record of i_2.



OSCILLOGRAM 9.

Primary and secondary currents in the $R_1L_1C_1M\ R_2L_2C_2$ combination.

$E = 30$ volts. $R_1 = 3.5$ ohms. $L_1 = 0.093$ henry. $C_1 = 150\mu$f.
$M = 0.026$ henry. $R_2 = 0.8$ ohm. $L_2 = 0.011$ henry. $C_2 = 168\mu$f.

Degree of Coupling.—To show the effect of various degrees of coupling, the determinantal equation,

$$Z(p) = p^4 + K_1p^3 + K_2p^2 + K_3p + K_4 = 0$$

was solved for two other values of M. When M is 0.02 henry, the roots of the equation (all other constants of the circuit remaining as before) are as follows:

$$p_1 = -\ 74.1 + j980 \qquad p_3 = -\ 16.65 + j256$$

$$p_2 = -\ 74.1 - j980 \qquad p_4 = -\ 16.65 - j256$$

When the value of $M = 0.01$ henry, the roots are:

$$p_1 = -42.8 + j775 \qquad p_3 = -18.45 + j265$$

$$p_2 = -42.8 - j775 \qquad p_4 = -18.45 - j265$$

The results of various degrees of coupling upon the damping factors and the frequencies of the components can best be shown by means of the following table.

M (Henrys)	Natural Frequencies (Cycles per Second)		Damping Constants	
0 026	40 8	208	16 1	146 9
0 020	41 0	156	16 65	74 1
0 010	42 2	123	18 45	42 8
0 000	42 5	117	18 8	36 4

From these results, it is evident that the degree of coupling for this particular circuit plays an important part in determining the behavior of the currents. In general, the natural frequencies of the circuits tend to move farther apart with tighter coupling, i.e., the circuit having the lower natural uncoupled frequency acquires one of still lower frequency, while the circuit having the higher natural uncoupled frequency acquires one of still higher frequency. Both frequencies are, of course, present in each circuit when they are coupled. With closer coupling, the lower-frequency component is not damped out as rapidly whereas the higher-frequency component is damped out much more rapidly. It should be recognized that the above statements are applicable only to the particular case under discussion, namely, a low-frequency primary and a high-frequency secondary.

Tuned Coupled Circuits.—In the preceding analysis the parameters in the primary and secondary circuits were such as to make the uncoupled natural frequencies of the individual loops distinctly different from one another. If $R_1 = R_2 = R$, $L_1 = L_2 = L$, and $C_1 = C_2 = C$, the individual loops will have the same uncoupled natural frequencies and under these conditions the mathematical analysis is somewhat simplified. The uncoupled natural frequency and damping constant of each circuit will be:

$$f' = \frac{\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}{2\pi} \tag{101}$$

and

$$\text{D.C.}' = -\frac{R}{2L} \tag{102}$$

When the circuits are magnetically coupled the operational expressions for i_1 and i_2 take the following forms:

$$i_1 = \frac{E(Lp^3 + Rp^2 + p/C)}{\left(Lp^2 + Rp + \frac{1}{C}\right)^2 - M^2p^4} \tag{103}$$

and

$$i_2 = \frac{-EMp^3}{\left(Lp^2 + Rp + \frac{1}{C}\right)^2 - M^2p^4} \tag{104}$$

In the above expressions the value of $Z(p)$ is common, namely,

$$Z(p) = \left(Lp^2 + Rp + \frac{1}{C}\right)^2 - M^2p^4$$

The right-hand member is readily recognized as the product of the sum and difference of two terms which are:

$$\left(Lp^2 + Rp + \frac{1}{C}\right) + Mp^2$$

and

$$\left(Lp^2 + Rp + \frac{1}{C}\right) - Mp^2$$

Setting these two factors equal to zero in turn and solving for p yields the roots of the expression $Z(p) = 0$. The roots are:

$$p_1, p_2 = \frac{-R}{2(L+M)} \pm j\sqrt{\frac{1}{(L+M)C} - \frac{R^2}{4(L+M)^2}} \tag{105}$$

$$p_3, p_4 = \frac{-R}{2(L-M)} \pm j\sqrt{\frac{1}{(L-M)C} - \frac{R^2}{4(L-M)^2}} \tag{106}$$

Circuit 1 and circuit 2, under the conditions of magnetic coupling, will have two natural frequencies, the angular velocities of which are defined by the radical terms of the above expressions. The manner in which

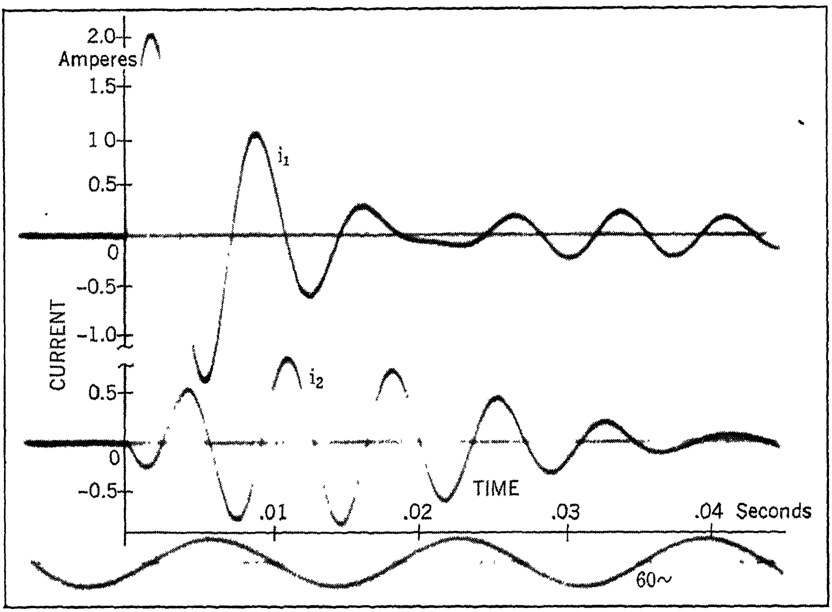
the magnetic coupling has affected the natural response of the circuits may be determined by comparing:

$$\sqrt{\frac{1}{(L+M)C} - \frac{R^2}{4(L+M)^2}} \quad \text{and} \quad \sqrt{\frac{1}{(L-M)C} - \frac{R^2}{4(L-M)^2}}$$

with:

$$\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

In general, one of the component frequencies under coupled conditions is lower and the other higher than the uncoupled natural frequency of



OSCILLOGRAM 10.

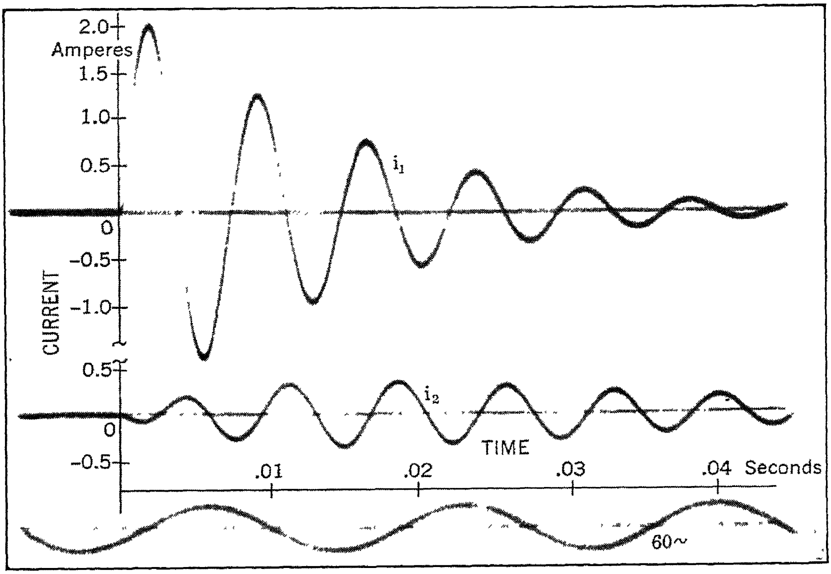
Primary and secondary currents in the $R_1L_1C_1M\ R_2L_2C_2$ combination.
$E = 30$ volts. $R_1 = R_2 = 3.2$ ohms. $L_1 = L_2 = 0.015$ henry.
$C_1 = C_2 = 83\mu$f. $M = 0.0028$ henry.

either loop. As M decreases, the component frequencies in i_1 and i_2 approach the uncoupled natural frequency of the individual circuits.

It is suggested that the student develop the expressions for i_1 and i_2 in the $R_1L_1C_1M\ R_2L_2C_2$ combination when $R_1 = R_2 = 3.2$ ohms, $L_1 = L_2 = 0.015$ henry, $C_1 = C_2 = 83$ microfarads, and $M = 0.0028$ henry. Oscillogram 10 shows the primary and secondary currents in a

combination having such parameters. The uncoupled natural frequency of either circuit is 141 cycles per second. Calculations will show that the frequencies of the component parts of i_1 and i_2 (coupled) are 130 and 157 cycles per second. A beat frequency of approximately 27 cycles per second may be observed on the oscillogram.

Oscillogram 11 illustrates the same phenomena as Oscillogram 10



Oscillogram 11.

Similar to Oscillogram 10 except for looser coupling between primary and secondary.

does except for looser coupling. M is sufficiently low in the present case to eliminate any apparent distortion of the i_1 and i_2 graphs.

EXERCISES

1. Check the numerical solution given in connection with the $R_1L_1M\ R_2L_2$ combination, and plot equations (23), (24), and (25). Compare the plotted solutions with Oscillogram 1.

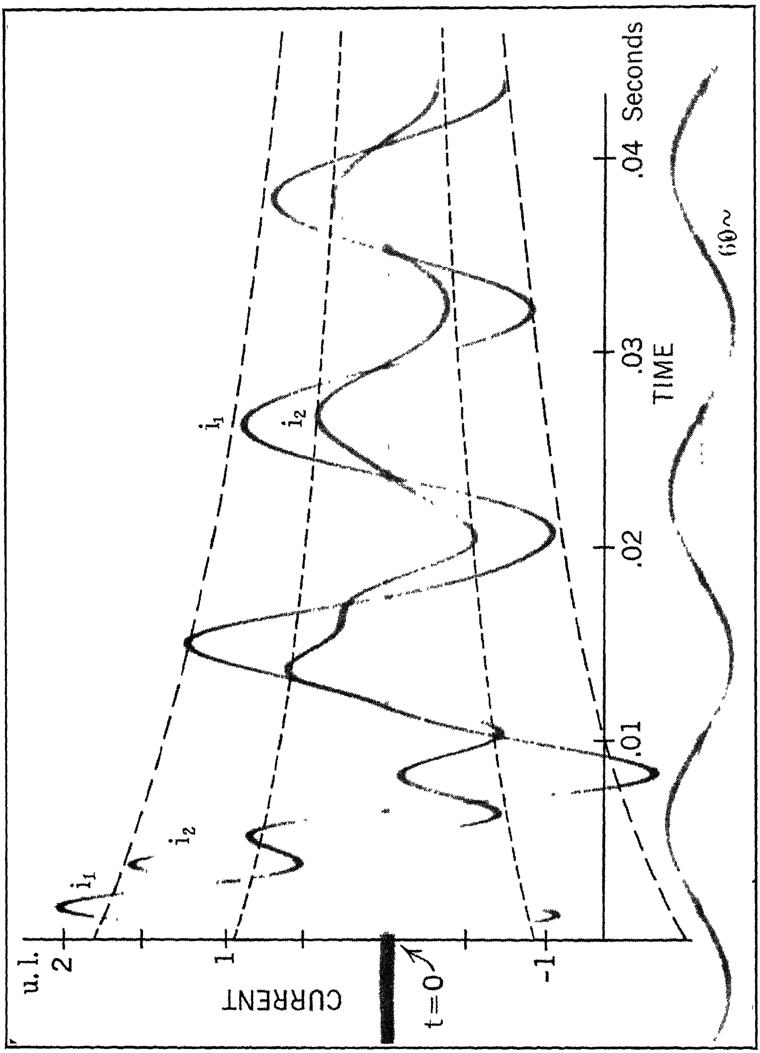
2. (*a*) Check the numerical solution given in connection with "Inductively Coupled Parallel Branches"

(*b*) Explain, physically, why i_1 (equation 53) becomes greater than E/R_1 during a certain interval in the transient period.

(*c*) Give a physical explanation of the negative current shown in Oscillogram 3.

3. Give physical explanations for the various trends of the i_1 curve shown in Oscillogram 4.

4. Derive the expressions for i_1 and i_2 of Fig. 6 under the condition that $R_1 = R_2 = 0$. Interpret the results.



Oscillogram 12.
The primary and secondary currents in an $R_1L_1C_1M$ $R_2L_2C_2$ combination. The oscillogram is to be used in connection with exercise No. 8.

5. Derive the expressions for i_1 and i_2 of Fig. 6 under the condition of perfect coupling, namely, $M = \sqrt{L_1L_2}$. Write the numerical expressions for i_1 and i_2 under the condition of perfect coupling when the circuit parameters, other than M, and the applied voltage are similar to those given in the numerical solution on page 127.

6. Refer to equations (99) and (100). Compare the relative magnitudes of the high- and low-frequency components at $t = 0.03$ second Check the numerical values obtained with the oscillographic records shown in Oscillogram 9.

7. It will be recalled that simple mechanical analogies were given in Chapter I for the circuit parameters R, L, and C. Can the concept of mutual induction be represented by a simple mechanical analogue? See: "A Mechanical Analogy for Coupled Electrical Circuits," by H. J. Reich, in The Review of Scientific Instruments, Vol. 3, No. 6, June, 1932.

8. Refer to Oscillogram 12. The graphs, i_1 and i_2, are the currents in the primary and the secondary respectively of an $R_1L_1C_1MR_2L_2C_2$ combination. The general expression for i_1 is given by equation (87); for i_2, by equation (92). The symbols referred to in the following questions apply to the symbols as used in equations (87) and (92).

(*a*) What are the approximate frequencies of the two component parts of i_1 and i_2? The determination is to be made directly from an inspection of the oscillogram.

(*b*) What are the approximate numerical values of b_1 and b_2? It is assumed that b_1 is the lower angular velocity of the two.

(*c*) What are the approximate numerical values of the damping constants, a_1 and a_2, as determined from the oscillogram?

(*d*) What is the physical significance of each of the constants: A, B, D, and F, in equations (87) and (92)?

(*e*) What are the approximate magnitudes of A, B, D, and F expressed in "unit lengths"? What are the signs of these quantities, assuming that σ_1, σ_2, δ_1, and δ_2 are expressed as less than π?

(*f*) What are the relative magnitudes and signs of σ_2 and δ_2, and of σ_1 and δ_1, that can be observed directly from the oscillographic records? How can the relative values of these angles be explained physically?

(*g*) Reconstruct the current-time graphs shown in Oscillogram 12 from the results of the foregoing analysis. Show each of the components that combine to form the resultant current graph.

9. Derive the general expressions for the subsidence of i_1 and i_2 in the R_1L_1M-R_2L_2 combination shown in Fig. 1 assuming that the battery is disconnected and that the R_1L_1 circuit is closed on itself at $t = 0$. It is further assumed that $i_1 = E/R_1$ and $i_2 = 0$ at $t = 0$.

SECTION II

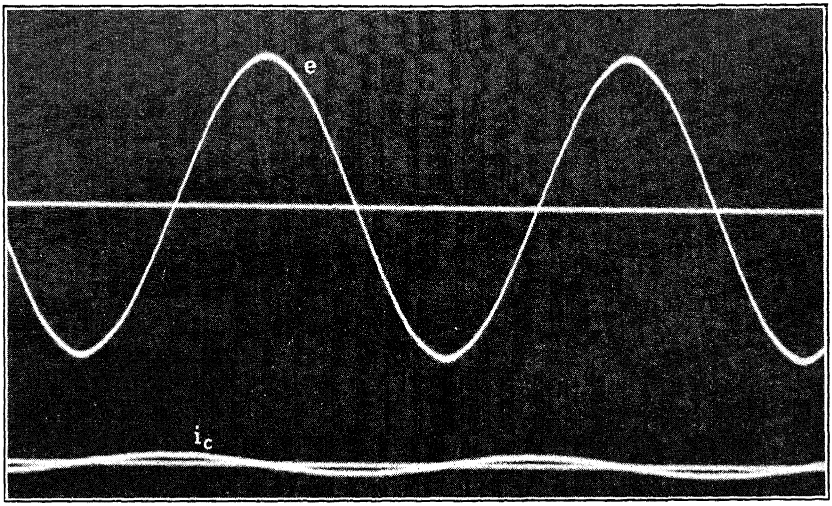
ALTERNATING-CURRENT TRANSIENTS

CHAPTER VI

ALTERNATING VOLTAGE APPLIED TO IDEAL CIRCUITS

Voltage Wave Form.—In actual practice, the vast majority of electrical circuits are energized by alternating potential differences. In general, the mode of alternation is approximately that of a sinusoidal variation with respect to time. The wave form of an alternating voltage is the shape of the curve obtained when the instantaneous values of voltage are plotted against time in rectangular coordinates. A wave form is sometimes said to be sinusoidal when the maximum difference between the ordinates of the wave form and the ordinates of an equivalent sine wave does not exceed a few per cent of the maximum value of the equivalent sine wave, assuming that the two curves are superimposed so as to make the differences as small as possible. The generation of a purely sinusoidal variation is, practically speaking, an impossibility, although Oscillogram 1 illustrates a very close approximation to such a wave form. It is only with specially constructed alternators and oscillators that such a close approximation to the sine wave can be obtained. A sinusoidal variation of potential difference is particularly desirable from the standpoint of transmission and utilization of electrical energy. Mathematically, the sine wave is the simplest with which to deal, and for this reason calculations are often based upon the assumption of sinusoidal variation when such is only approximately correct.

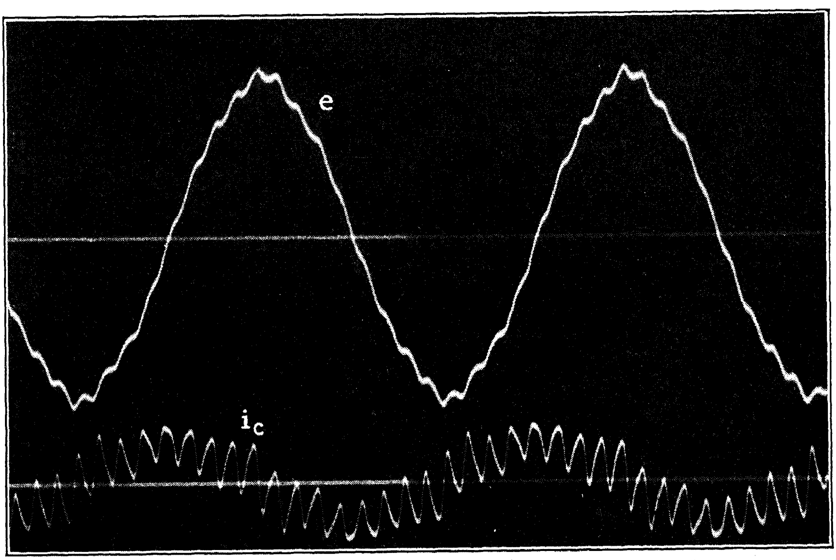
Occasionally wave forms of potential differences are encountered that differ materially from sine waves. The wave forms of the output voltage of overloaded vacuum-tube oscillators are very often distinctly non-sinusoidal. The so-called high-frequency alternators, as well as many of the low-frequency alternators of small kilovolt-ampere capacity, generate non-sinusoidal wave forms of potential difference. Oscillogram 2 illustrates the voltage wave form of a small laboratory alternator. The open type of slot combined with the shaping and dimensions of the pole face are responsible for the non-sinusoidal wave form shown. Inspection of Oscillogram 2 shows that a pronounced seven-



OSCILLOGRAM 1.

e = wave form of voltage generated by a sine wave alternator.

i_c = condenser current, wherein deviations from a sinuosid in the voltage wave appear in amplified form. The smoothness of this graph indicates the absence of higher harmonics in the voltage wave.



OSCILLOGRAM 2.

e = wave form of voltage generated by an open-slot type alternator of small kilovolt-ampere capacity.

i_c = condenser current. The seventeenth harmonic in the voltage variation produces a pronounced seventeenth harmonic in the i_c variation.

teenth harmonic is present in the voltage wave of this particular machine. An assumption of sinusoidal wave form in this case may lead to gross inaccuracies, particularly if capacitance effects are present.

The i_c graphs shown on Oscillograms 1, 2, and 3 are photographic records of the currents taken by condensers of small capacitance which have been placed across the alternator terminals. Any deviations from a true sinusoidal variation in the voltage wave form are shown in an amplified manner in the condenser currents. For example, let the voltage wave be represented by

$$e = E_{m1} \sin \omega t + E_{m5} \sin 5\omega t$$

The steady-state current through an ideal condenser branch to which the above voltage is applied may be written as follows:

$$i_c = C \frac{de_c}{dt}$$

$$i_c = \omega C E_{m1} \cos \omega t + 5\omega C E_{m5} \cos 5\omega t$$

The ratio of the maximum value of the fifth harmonic current to the maximum value of the fundamental component is:

$$\frac{I_{m5}}{I_{m1}} = \frac{5E_{m5}}{E_{m1}}$$

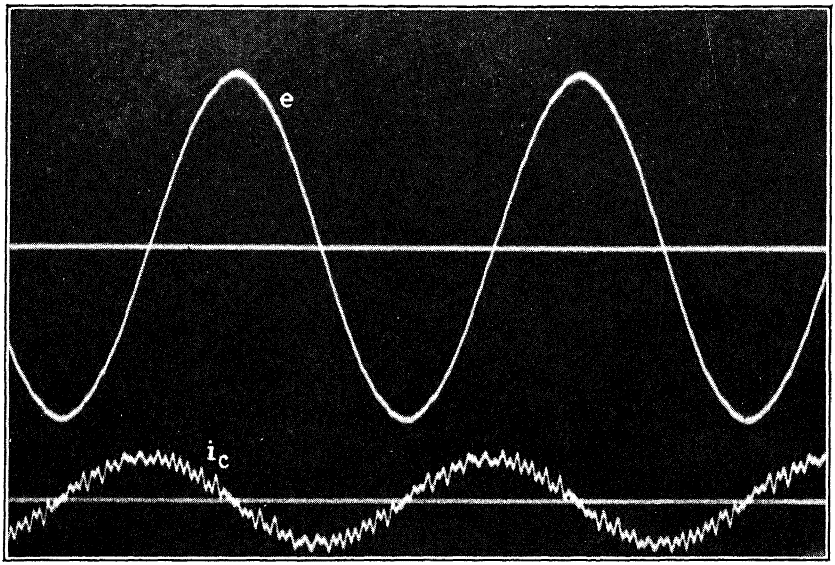
The ratio of the maximum values of the fifth harmonic voltage to its fundamental is $\frac{E_{m5}}{E_{m1}}$. Under these conditions a 20 per cent fifth harmonic component in the voltage wave will give rise to a fifth harmonic current whose maximum is equal in magnitude to the maximum of the fundamental current.

In Oscillogram 2, the maximum value of the seventeenth harmonic component in the current is approximately 50 per cent as large as the maximum value of the fundamental.

In general, the wave forms of large commercial alternators do not differ materially from a sinusoid. A typical commercial wave form is shown in Oscillogram 3. Except, possibly, in the calculation of condenser currents such a wave form may be treated as a sine wave, and analyses based upon that assumption will generally be within the limits of engineering accuracy. Sinusoidal variations of potential difference are assumed in the mathematical discussions of the following chapters, and all oscillograms that are shown have been taken under conditions that closely approximate the sine wave form of applied emf. This should place no serious limitation on the mathematical analyses inas-

much as the separation of non-sinusoidal wave forms into sinusoidal components is a simple, although somewhat tedious, task.

If the wave form of the voltage applied to an electric circuit deviates materially from a pure sinusoidal variation, the resultant wave may be separated into sinusoidal components of different frequencies and, in general, of different magnitudes. The current *due to* each of the sinusoidal components of applied voltage may then be evaluated by ordinary methods, provided that the circuit parameters are constant. The



OSCILLOGRAM 3.

e = wave form of voltage generated by an ordinary commercial alternator.
i_c = condenser current.

actual current response may be obtained by combining the component currents since the current solutions, thus obtained, are superimposable.

Nomenclature.—When dealing with alternating potential differences an exact understanding of the nomenclature is quite necessary. Throughout the remainder of this text the following notation is employed:

t = time, the independent variable, reckoned from a specified $t = 0$.
e = the instantaneous value of the applied emf.
E_m = the maximum value of e.
f = the frequency of the variation of e, expressed in cycles per second.

$\omega = 2\pi f$. ω is the factor by which time is multiplied to obtain angular measure. As such it is an angular velocity.

ωt represents the angular displacement that takes place along the wave variation between the times $t = 0$ and any later time, t.

λ is the angular displacement expressed in degrees or radians, between $e = 0$ and $t = 0$ measured positively from $e = 0$. The $e = 0$ point referred to is the zero point at which $\frac{de}{dt}$ is positive. Fig. 1 will show the meaning of λ more clearly.

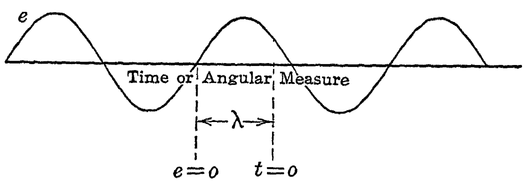


FIG. 1.—λ indicates the point on the voltage wave at which $t = 0$.

The general expression for sinusoidal applied voltage is:

$$e = E_m \sin (\omega t + \lambda)$$

e varies sinusoidally, reaching maximum and minimum values of $+E_m$ and $-E_m$ respectively. The frequency of variation is $\omega/(2\pi)$ cycles per second provided, of course, that ω is expressed in radians per second. The factor λ permits the reckoning of time from any point along the sine wave. To count time from, say, a positive maximum, λ is set equal to $+\pi/2$. If it is desired to count time from a point on the voltage wave $\frac{1}{6}$ of a cycle prior to e reaching its zero value, $\left(\frac{de}{dt} \text{ positive}\right)$, λ is given the value of $-\pi/3$.

The R Circuit.—A diagram of a purely resistive circuit to which a sinusoidal exciting voltage may be applied is shown schematically in Fig. 2. Assuming that the switch is closed at $t = 0$, the condition for dynamic equilibrium in the circuit is:

$$Ri = E_m \sin (\omega t + \lambda) \tag{1}$$

from which:

$$i = \frac{E_m}{R} \sin (\omega t + \lambda) \tag{2}$$

At $t = 0$ the exciting voltage is equal to $E_m \sin \lambda$, and the current instantly acquires a value equal to $(E_m/R) \sin \lambda$. The time at which

the switch is closed is, therefore, a most important factor in determining the initial current. If the switch is closed at the time e is equal to zero ($\lambda = 0$ or $\lambda = \pi$) the initial current is zero. If the switch is closed during the positive half of the cycle of the e variation, the initial current is a definite positive value; and if the switch is closed during the negative half of the cycle, the initial current at once acquires its negative steady-state value.

Assuming R to be constant, the current variation in the circuit after $t = 0$ is of exactly the same nature as the e variation. Inasmuch as equation (2) represents the complete solution of i, no transient effects are present in the ideal R circuit. The current instantly acquires its steady-state value at $t = 0$ and thereafter varies sinusoidally in time

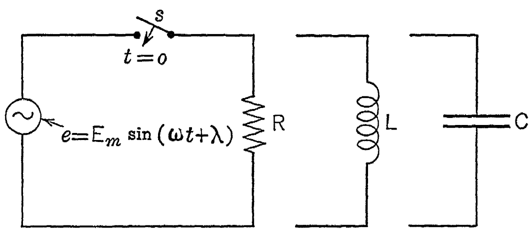


FIG. 2.—Ideal R, L, and C branches to which a sinusoidal variation of potential difference is applied at $t = 0$.

phase with the exciting voltage. Oscillograms 4 and 5 illustrate the effect of suddenly applying a 60-cycle sinusoidal emf to a constant resistance. In Oscillogram 4 the switch is closed at $e = 0$, and in Oscillogram 5 the circuit is completed at the time of maximum e. The effect of the inherent self-inductance of the resistance is discernible in Oscillogram 5.

The L Circuit.—If the resistance of Fig. 2 is replaced by a purely inductive element, dynamic equilibrium obtains when:

$$L\frac{di}{dt} = E_m \sin(\omega t + \lambda) \tag{3}$$

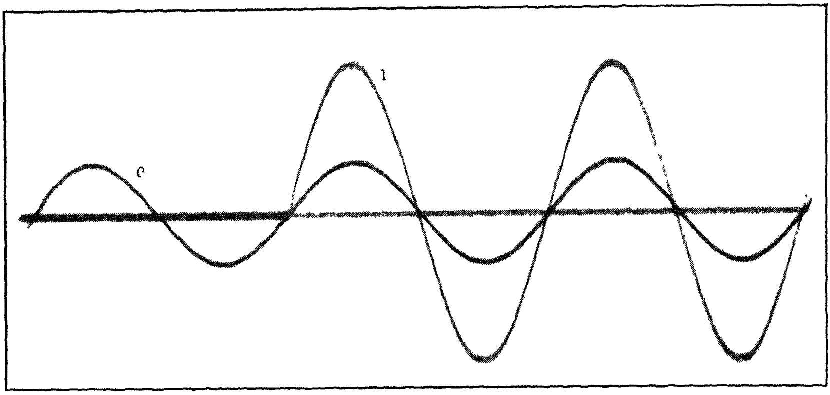
The counter-voltage of self-inductance must at all times equal the applied emf.

Separating variables in equation (3) thus:

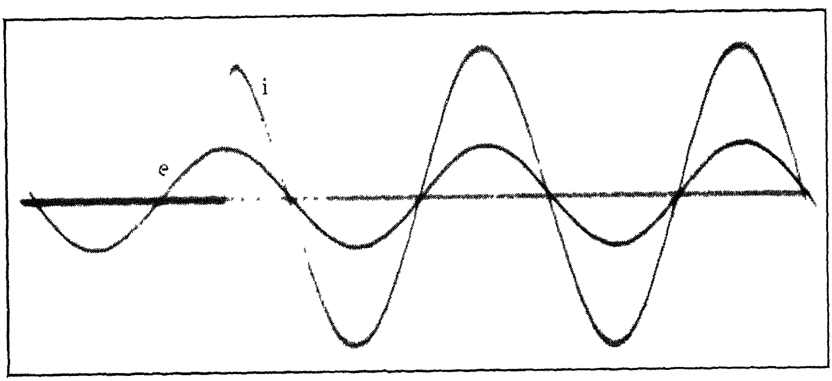
$$di = \frac{E_m}{L}\sin(\omega t + \lambda)dt$$

and integrating, gives:

$$i = -\frac{E_m}{\omega L}\cos(\omega t + \lambda) + c \tag{4}$$



OSCILLOGRAM 4.
e = sinusoidal applied emf, 60 cycles, 110 volts effective.
i = initial current in a resistance circuit, $\lambda = 0$.



OSCILLOGRAM 5.
e = sinusoidal applied emf, 60 cycles, 110 volts effective.
i = initial current in a resistance circuit, $\lambda = \pi/2$.

The electrical inertia of the circuit requires that $i = 0$ at $t = 0$. Hence:

$$c = \frac{E_m}{\omega L} \cos \lambda \tag{5}$$

and:

$$i = \frac{E_m}{\omega L} [\cos \lambda - \cos (\omega t + \lambda)] \tag{6}$$

The first term within the brackets represents the transient term, and the second is the steady-state component of the current. In the case of the ideal L circuit the transient term persists indefinitely, a phenomenon not encountered in actual circuits on account of the resistance

which is always present. For any given time of closing the switch, λ has a particular value and $\cos \lambda$ is a constant. The current is, therefore, a negative cosine variation about the $[\cos \lambda]$ ordinate.

If the switch is closed at $e = 0 \left(\frac{de}{dt} \text{ positive}\right)$ λ is zero and the current variation is:

$$i = \frac{E_m}{\omega L}[1 - \cos \omega t] \tag{7}$$

Equation (7) represents a minus cosine wave displaced positively so that i starts at zero at $t = 0$. For $\lambda = 0$ the displacement given by the transient term is maximum and is equal to the maximum value of the steady-state current. The resultant current, under these conditions, periodically acquires values that are twice as large as the maximum steady-state current. A graphical representation of equation (7) is given in Fig. 3.

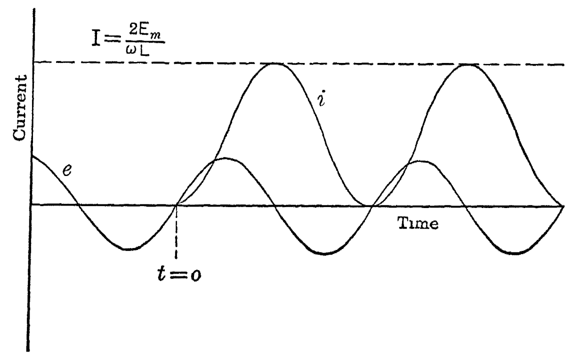


FIG. 3.—Current-time graph of a purely inductive circuit when a sinusoidal voltage is applied. $e = 0$ at $t = 0$.

The current variation in a highly inductive circuit for the case of $\lambda = 0$ is illustrated photographically in Oscillogram 6. The resistance of the circuit is, obviously, responsible for the slight departure of the current variation from the $[1 - \cos \omega t]$ form.

Closing the switch at the time of maximum voltage results in complete disappearance of the transient term. λ is equal to $\pi/2$, and $\cos \lambda$ is equal to zero. The expression for the current variation then becomes:

$$i = -\frac{E_m}{\omega L}\cos\left(\omega t + \frac{\pi}{2}\right) \tag{8}$$

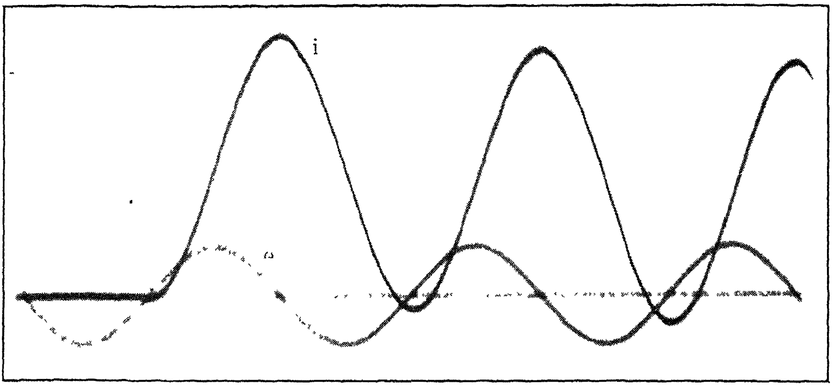
or

$$i = \frac{E_m}{\omega L}\sin \omega t \tag{9}$$

The time phase of i with respect to e is apparent considering that:

$$e = E_m \sin\left(\omega t + \frac{\pi}{2}\right) \tag{10}$$

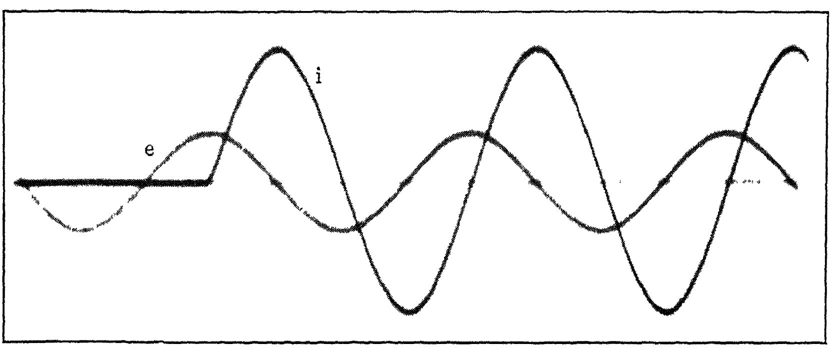
The current starts at zero value and continues, thereafter, as a sinusoidal variation lagging the applied voltage by a quarter of a cycle. Oscillogram 7 illustrates the current and voltage waves when $\lambda = \pi/2$.



OSCILLOGRAM 6.

e = applied emf, 60 cycles, 110 volts effective.
i = initial current in a highly inductive circuit, $\lambda = 0$.

In a purely inductive circuit the rate of change of current with respect to time is positive during that portion of the cycle in which the applied voltage is positive and it is negative during that portion



OSCILLOGRAM 7.

e = applied emf, 60 cycles, 110 volts effective.
i = initial current in a highly inductive circuit, $\lambda = \pi/2$.

of the cycle in which the applied voltage is negative. As contrasted to the application of a constant potential difference, the application of an alternating potential difference to a purely inductive circuit limits the current to definite maximum and minimum values. Under

no condition can the current become greater than $\frac{2E_m}{\omega L}$ and in case $\lambda = \pm \pi/2$ the current is limited to $\frac{E_m}{\omega L}$. ωL is the total impedance of the ideal L circuit. It is commonly called the inductive reactance, and as such it appears as one component in the impedance function of actual circuits.

The C Circuit.—The consideration of a purely capacitive circuit is, in some respects, more of a hypothetical venture than were the considerations of purely resistive and purely inductive circuits. Let R of Fig. 2 be replaced by a perfect condenser of a given capacitance, C, and assume that the condenser is without charge prior to $t = 0$. If, at the time the switch is closed, e has a magnitude other than zero, the condenser instantly acquires a charge such that $q/C = e$. In this speculative case the current would necessarily be of infinite magnitude at $t = 0$. As long thereafter as the switch remains closed the condition for circuit equilibrium is:

$$\frac{q}{C} = E_m \sin(\omega t + \lambda) \tag{11}$$

Differentiating gives:

$$\frac{dq}{dt} = E_m \omega C \cos(\omega t + \lambda)$$

from which:

$$i = \frac{E_m}{\frac{1}{\omega C}} \cos(\omega t + \lambda) \tag{12}$$

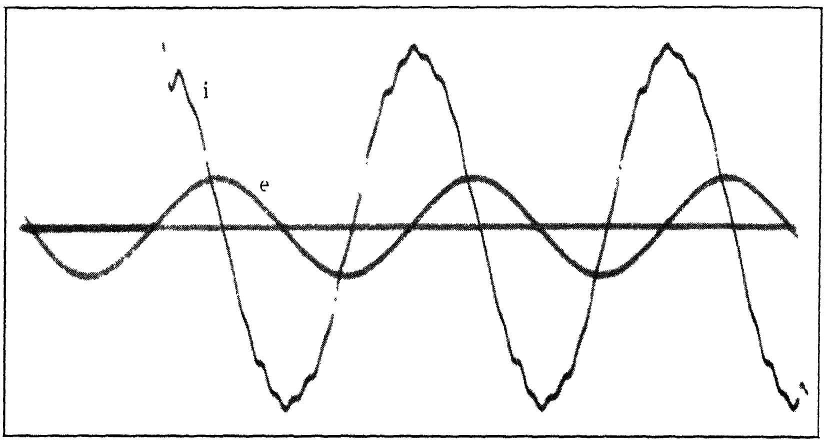
It is apparent that equation (12) is not the complete expression for current in the circuit. More elaborate schemes of notation are required if the infinite current at $t = 0$ is to be included. Equation (12) represents only the steady-state term of the complete expression for i. The transient term, if present, is an additional factor that is infinitely large for an infinitely short period of time. It is by virtue of this initial current inrush that the condenser acquires the charge necessary for dynamic equilibrium at $t = 0$.

If the switch is closed at $\lambda = 0$ no transient current is required and the complete expression for current is

$$i = \frac{E_m}{\frac{1}{\omega C}} \cos(\omega t) \tag{13}$$

At $t = 0$ the current instantly rises to its positive maximum value and thereafter becomes a cosinusoidal variation, 90 degrees in advance of e. The impedance of the pure C circuit under steady state-conditions is $\frac{1}{\omega C}$.

Oscillogram 8 illustrates the behavior of a circuit that approximates an ideal C circuit. The switch is closed at approximately $e = 0$ $\left(\frac{de}{dt}\text{ positive}\right)$ with the result that the current very quickly reaches its $+ E_m\omega C$ value and thereafter continues its fundamental 60-cycle variation a quarter of a cycle ahead of the applied voltage. The small irregu-



OSCILLOGRAM 8.

e = applied emf, 60 cycles, 110 volts effective.

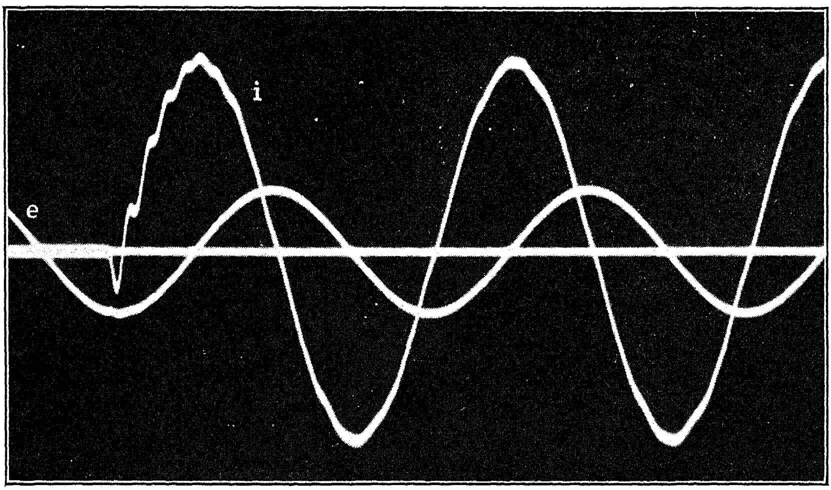
i = initial current in a highly capacitive circuit, $\lambda = 0$.

larities in the applied voltage form appear in an exaggerated manner in the wave form of the condenser current because the instantaneous value of the current is dependent upon the rate of change of voltage.

It is, obviously, impossible with an electromagnetic type of oscillograph to show the exact behavior of the circuit if λ is other than zero or π. The initial inrush of current is enormous if the switch is closed at a time such that an appreciable voltage is applied to the circuit at $t = 0$.

The matter of initial charge must be considered whenever condensers are present in a circuit. Most dielectrics are capable of maintaining electric charges on the plates of a condenser over considerable periods of time. If the condenser has been in an active circuit it will, in general, be left in a charged condition. The latter statement carries with it the

assumption that no path is provided for discharge after the circuit has been rendered inactive. A potential difference equal to Q_0/C exists between the terminals of the condenser under these conditions. When such a condenser again becomes a part of a closed circuit this voltage exhibits itself as one of the component voltages. The sign of the quantity, Q_0, as used in mathematical expressions is entirely a matter of convention. If Q_0/C is considered to be a driving voltage, Q_0 has a definite sign, either positive or negative, in a particular case. Considered as a counter-voltage, the same Q_0 is opposite in sign. In terms of



OSCILLOGRAM 9.

e = applied emf, 60 cycles, 110 volts effective.

i = initial current in a highly capacitive circuit, illustrating the stabilizing effect of Q_0.

its counter-voltage effect, Q_0 is positive if Q_0/C acts opposite to a positive driving voltage.

The initial condenser voltage may cause serious disturbances in the circuit momentarily. It may, of course, with the opposite polarity be a stabilizing factor. An example of this is shown in Oscillogram 9. The conditions under which the oscillogram was taken approximate, roughly, the hypothetical case which is being discussed, namely, a purely capacitive circuit. It will be observed that the circuit becomes active at the point of approximately maximum applied voltage. If the condenser had been discharged prior to the time of closing the switch a large inrush of current would have resulted. The magnitude, polarity, and general effect of the initial condenser charge in this case are quite obvious. The irregularities in the current variation near $t = 0$ are the

result of small transient oscillations, caused by energy interchanges between the electrostatic field of the condenser and the electromagnetic field of the armature winding of the alternator.

EXERCISES

1. A 60-cycle alternating potential difference of 110 volts (effective) is applied to a 10-ohm resistance at $t = 0$ The switch is closed 0.005 second after the beginning of a cycle of the potential variation. Assuming sinusoidal variation of e, write the expression for the current variation employing numerical coefficients. What is the value of the initial current? What is the maximum value of the current?

2. Repeat the above exercise assuming that the 10-ohm resistance is replaced by a coil of 0 04-henry inductance. Neglect the resistance of the inductance coil.

3. A 60-cycle alternating potential difference of 110 volts (effective) is applied to a 200-microfarad condenser at the time of zero emf. What is the value of the initial current? Write the equation for i in terms of time employing numerical coefficients. Assume that $Q_0 = 0$.

4. (*a*) Discuss the effect of closing the switch in problem 3 at the point of maximum emf. Assuming that the charge required for equilibrium is transferred at a uniform rate to the condenser in 0.0001 second, what value of current flows in the circuit during that period?

(*b*) What magnitude and polarity of Q_0 will cause the current to be zero at $t = 0$?

5. Compare the dimensions of ωL and $\frac{1}{\omega C}$ with the dimensions of R. Employ the L–q–t system which is outlined in tabular form on page 12

6. It will be recognized that quantities must have similar dimensions before they can be added. Does it follow that, because physical quantities have the same dimensions, they can be added? Extend the argument to include R, ωL, and $\frac{1}{\omega C}$.

7. Write the equations for the following 60-cycle alternating voltages with respect to the indicated $t = 0$ points.

$$(a)\quad e_0 = +\,135 \text{ volts}, \ \frac{de}{dt} \text{ positive.}$$

$$(b)\quad e_0 = -\,135 \text{ volts}, \ \frac{de}{dt} \text{ negative.}$$

$$(c)\quad e_0 = +\,110 \text{ volts}, \ \frac{de}{dt} \text{ negative.}$$

$$(d)\quad e_0 = -\,110 \text{ volts}, \ \frac{de}{dt} \text{ positive.}$$

$e_0 = e$, at $t = 0$. The effective value of each of the voltage waves is 110 volts. Write the equations in a manner such that, in each case, $\lambda < 90°$.

CHAPTER VII

TRANSIENTS IN SERIES CIRCUITS WITH SINUSOIDAL IMPRESSED VOLTAGE

THE *RL* CIRCUIT

In actual practice, series circuits consisting of resistance and self-inductance are very common. Their behavior to the sudden application of a sinusoidal variation of potential difference is, therefore, of much greater importance than was the behavior of the ideal circuits which were considered in the previous chapter.

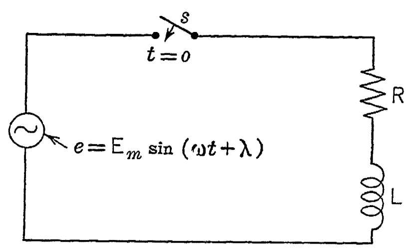


FIG. 1.—The *RL* circuit to which a sinusoidal voltage is applied at $t = 0$.

Physical Considerations.—With the circuit arrangement as shown in Fig. 1, the condition for dynamic equilibrium at the time of and after closing switch, s, is that the applied emf must be balanced by the counter-voltages, Ri and $L\frac{di}{dt}$. This condition may be written in equation form as follows:

$$Ri + L\frac{di}{dt} = E_m \sin(\omega t + \lambda) \tag{1}$$

It is known, from elementary alternating-current theory, that the "steady-state" value of the current is of the form:

$$i_s = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) \tag{2}$$

where

$Z = \sqrt{R^2 + \overline{\omega L}^2}$, the steady-state impedance.

$\theta = \tan^{-1}\frac{\omega L}{R}$, the angle of lag of the steady-state component of the current with respect to the applied voltage.

By steady-state alternating-current values are meant those which are recurring periodic functions of time. The steady-state term, i_s, persists in the circuit as long as the switch remains closed. Though there are more direct methods of attack, it is only necessary to substitute the value of i_s as given by equation (2) in equation (1) to prove that $(E_m/Z)\sin(\omega t + \lambda - \theta)$ is the correct expression for the steady-state term of the current. Thus:

$$Ri_s + L\frac{di_s}{dt} = E_m \sin(\omega t + \lambda). \tag{1a}$$

Substituting in the left-hand member gives:

$$R\left[\frac{E_m}{Z}\sin(\omega t + \lambda - \theta)\right] + L\left[\frac{\omega E_m}{Z}\cos(\omega t + \lambda - \theta)\right]$$

Substituting $\cos\theta$ for R/Z and $\sin\theta$ for $\omega L/Z$ yields:

$$E_m\{[\sin(\omega t + \lambda - \theta)\cos\theta] + [\cos(\omega t + \lambda - \theta)\sin\theta]\}$$

The above bracket expression will be recognized as the expanded form of the $\sin(\alpha + \beta)$ where $\alpha = (\omega t + \lambda - \theta)$ and $\beta = \theta$. Therefore the left-hand member of equation (1a) reduces to

$$E_m \sin(\omega t + \lambda - \theta + \theta)$$

which is equal to the right-hand member.

But a consideration of the physical facts involved will show that i_s cannot be the complete expression for the current. The self-inductance of the circuit requires that the actual current be zero at $t = 0$. Except for the case of $(\lambda - \theta) = 0$ or π, i_s has a value other than zero at $t = 0$. Therefore, another component of current which is equal in magnitude but opposite in direction to the initial i_s must be present in the circuit at $t = 0$. In general a certain period of time elapses before the current is completely expressed by i_s, and it is during this period that the transient behavior of the circuit is in evidence.

Let the transient component of the current be known as i_t; then the actual current in the circuit may be written as follows:

$$i = i_s + i_t \tag{3}$$

Substituting the value $(i_s + i_t)$ for i in equation (1) yields:

$$Ri_t + L\frac{di_t}{dt} = 0 \tag{4}$$

Since i_t is a part of the complete expression for i, its evaluation in terms

of the circuit parameters is desirable. Mathematically, equation (4) is known as the complementary function. Physically it states the relationship that must exist between Ri_t and $L\frac{di_t}{dt}$. The same relationship was encountered when the decay current in an RL circuit was considered in Chapter III. i_t starts, at $t = 0$, with a value equal and opposite to the initial i_s and diminishes exponentially to zero.

The solution of equation (4) for i_t is of the form:

$$i_t = A\epsilon^{-\frac{Rt}{L}} \tag{5}$$

where A is the constant of integration and must be determined from the boundary condition, namely, that $i = 0$ at $t = 0$.

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + A\epsilon^{-\frac{Rt}{L}} \tag{6}$$

At $t = 0$:

$$i = \frac{E_m}{Z}\sin(\lambda - \theta) + A = 0$$

from which:

$$A = -\frac{E_m}{Z}\sin(\lambda - \theta) \tag{7}$$

The complete expression for the current in terms of the applied voltage and the circuit parameters thus becomes:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) - \frac{E_m}{Z}\sin(\lambda - \theta)\epsilon^{-\frac{Rt}{L}} \tag{8}$$

It will be observed that equation (8), if substituted in equation (1), satisfies the condition for dynamic equilibrium. Furthermore, equation (8) shows that the transient term is equal in magnitude but opposite in sign to the steady-state term at $t = 0$. Of course, equation (8) should yield this information since it was upon these bases that the expression for i was obtained. Such interpretation of mathematical expressions, however, is a most desirable means of becoming acquainted with their real meaning.

If the switch is closed at a point on the applied voltage wave so that $(\lambda - \theta) = 0$, or $(\lambda - \theta) = \pi$, there will be no transient term. At these points i_s is equal to zero. Under these conditions the expression for the steady-state component represents the complete expression for the current in the circuit, and no transient component of current is required. Fig. 2 shows the condition in a graphical manner.

The transient component may have any value between the negative maximum and the positive maximum of i_s, at $t = 0$. In general the condition that produces the largest possible transient term is of most importance because it is that value which must be reckoned with in case the switch is closed at random. In practice it is customary to operate RL circuits with no means of controlling the point on the applied voltage wave at which the switch is closed or opened. In the case of transmission lines and feeder circuits, the circuit-breaker capacity is determined on the basis of short circuits occurring at the most inopportune point of the voltage variation, namely, the point that will cause the greatest transient disturbance.

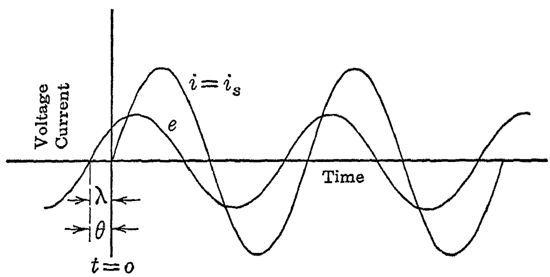


FIG. 2.—Graph of equation (8) for the case of $\lambda = \theta$. Under these conditions the transient component of the current in the RL circuit is equal to zero.

For given values of R and L, the largest transient term appears when:

$$(\lambda - \theta) = \frac{n\pi}{2}$$

where n is any positive odd integer.

If ωL is large as compared with R, θ will approach $\pi/2$, and the largest transient occurs when λ is close to zero or 180 degrees. If ωL is small as compared with R, θ will approach zero and the transient term will be greatest when λ is close to $\pi/2$ or $3\pi/2$. Plotted graphs are shown in Fig. 3 for the case of closing the switch at $\lambda = +350°$ or $-10°$. The term "closing the switch at $\lambda = 350°$" is merely a convenient manner of expressing the time. If the voltage variation is that of a 60-cycle alternator this is equivalent to closing the switch at 0.0162 second after the beginning of a cycle. The power factor angle of the circuit, θ, is taken as 80 degrees in this particular case. It will be remembered that:

$$\theta = \tan^{-1} \frac{\omega L}{R}$$

Therefore the ratio $\omega L/R$ for the case being considered is 5.67. With $\lambda = 350°$ and $\theta = 80°$:

$$\sin(\lambda - \theta) = -1$$

Under these conditions the largest possible transient term appears. Fig. 3 shows i_s, i_t, and i together with the impressed voltage wave. It will be noted that the maximum possible instantaneous current is somewhat less than twice the maximum value of the steady-state component.

Mathematical Analysis.—The method that has been employed in arriving at equation (8) lends itself nicely to physical interpretations,

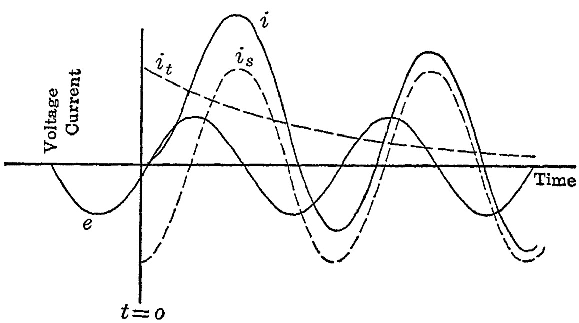


FIG. 3.—Graph of equation (8) for the case of $\sin(\lambda - \theta) = -1$. Under these conditions the transient component of the current in the *RL* circuit is of maximum value.

but more direct methods of attack are available. Various schemes have been devised for solving equation (1), some of which are very awkward because of the tedious integration that is involved. The latter may be eliminated largely by resorting to vector form of notation. Since a general understanding of the vector calculus on the part of the reader is not assumed, a conventional type of solution is outlined below.

The Conventional Solution.—Arranged in a more typical form, equation (1) takes the form:

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_m}{L} \sin(\omega t + \lambda) \tag{9}$$

Multiplying both sides of the equation by $\epsilon^{\frac{Rt}{L}}$ reduces the left-hand member to a single derivative form.

$$\epsilon^{\frac{Rt}{L}} \frac{di}{dt} + \frac{R}{L} \epsilon^{\frac{Rt}{L}} i = \frac{E_m}{L} \epsilon^{\frac{Rt}{L}} \sin(\omega t + \lambda) \tag{10}$$

Let

$$u = i\epsilon^{\frac{Rt}{L}} \tag{11}$$

Then

$$\frac{du}{dt} = \frac{E_m}{L}\epsilon^{\frac{Rt}{L}} \sin(\omega t + \lambda) \tag{12}$$

since

$$\frac{du}{dt} = \epsilon^{\frac{Rt}{L}}\frac{di}{dt} + \frac{R}{L}\epsilon^{\frac{Rt}{L}} i$$

$$u = \frac{E_m}{L}\int \epsilon^{\frac{Rt}{L}} \sin(\omega t + \lambda)dt \tag{13}$$

The above integral may be evaluated by the method of successive parts, but it is somewhat more convenient to change to exponential form prior to the actual integration. By either method it may be shown that:

$$u = \frac{E_m}{Z}\epsilon^{\frac{Rt}{L}} \sin(\omega t + \lambda - \theta) + c_1 \tag{14}$$

Therefore:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + c_1\epsilon^{-\frac{Rt}{L}} \tag{15}$$

Since

$$i = 0 \text{ at } t = 0:$$

$$c_1 = -\frac{E_m}{Z}\sin(\lambda - \theta) \tag{16}$$

and

$$i = \frac{E_m}{Z}\left[\sin(\omega t + \lambda - \theta) - \epsilon^{-\frac{Rt}{L}}\sin(\lambda - \theta)\right] \tag{17}$$

The Operational Solution.—Equation (1) takes the following operational form:

$$i(R + Lp) = E_m \sin(\omega t + \lambda) \tag{18}$$

$$i = \frac{E_m \sin(\omega t + \lambda)}{(R + Lp)} \tag{19}$$

Letting

$$a = \frac{R}{L}$$

$$i = \frac{E_m}{L}\frac{\sin(\omega t + \lambda)}{(a + p)} \tag{20}$$

The operational solution requires a knowledge of the meaning of

$$\frac{1}{a + p} \text{ operating on } \sin(\omega t + \lambda)$$

In the Appendix, page 288, it is shown that:

$$\frac{\sin(\omega t + \lambda)}{a + p} = \frac{1}{\sqrt{a^2 + \omega^2}} [\sin(\omega t + \lambda - \theta) - \epsilon^{-at} \sin(\lambda - \theta)]$$

where

$$\theta = \tan^{-1} \frac{\omega}{a}$$

It follows that

$$i = \frac{E_m}{Z} [\sin(\omega t + \lambda - \theta) - \epsilon^{-at} \sin(\lambda - \theta)] \tag{21}$$

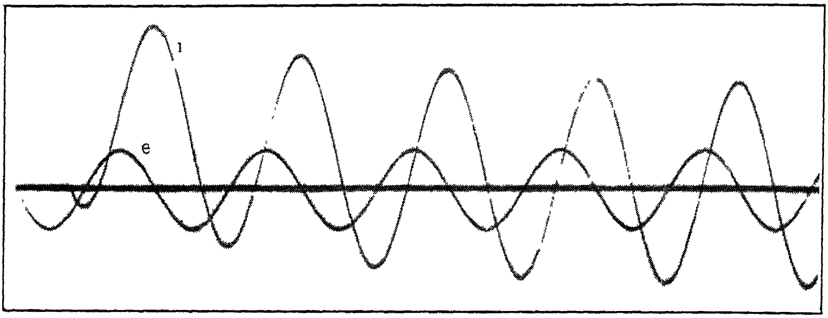
The above is typical of one class of operational solutions. After solving the fundamental equation explicitly for the dependent variable the expression is arranged by algebraic manipulations into type form The mathematical significance of a large number of typical operational forms has been determined although, in simple cases, it is often more difficult to make these determinations than it is to solve the problem by other methods.

Oscillographic Verification.—Oscillograms 1 and 2 show the starting currents in a highly inductive circuit. Comparison illustrates the effect of closing the switch at different points of the cycle. An examination of the gradual manner in which the current approaches its steady-state magnitude and phase position may be of interest. The circuit arrangement is similar to that shown in Fig. 1 except for the insertion of the voltage and current galvanometers.

Power Transients.—The current with respect to time and therefore with respect to the applied voltage having been determined, the information necessary for a study and analysis of power transients is available. The power is, at any time, the product of the instantaneous value of the applied voltage and the instantaneous value of the current. As such it is an important factor in determining the size of the switch gear that controls the circuit.

The expression for the instantaneous power in an alternating-current circuit is, in general, rather awkward to write in trigonometric form. For example, multiplying equation (8) by $[E_m \sin(\omega t + \lambda)]$ results in a form that is cumbersome and which admits of little reduction or simplification. But it is really not important that the equation be written, as the nature of the power variation may be determined from the plotted solution of the instantaneous power represented by the product of the instantaneous voltage and current. With the aid of

the watt-galvanometer, power transients can easily be made available for analysis, and when the frequency is well below the natural frequency

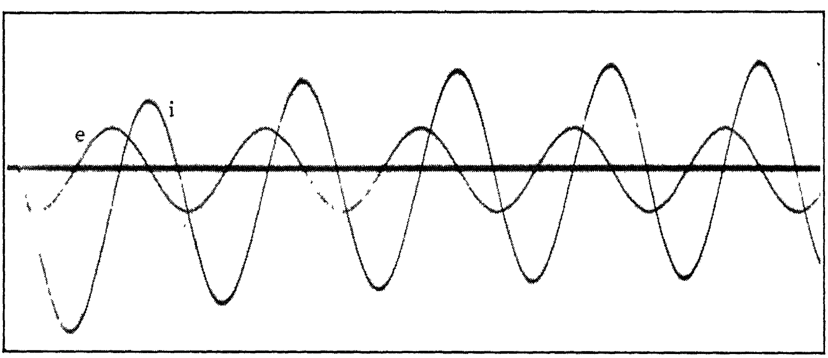


Oscillogram 1.

Transient current in the RL circuit.

e = 60-cycle applied emf. 115 volts, effective.
i = current variation.
L = 0 092 henry. R = 3.5 ohms.
λ = − 40°, approximately.

of the element (3600 cycles per second in the modern type of galvanometer) a fair degree of accuracy can be obtained.



Oscillogram 2.

Transient current in the RL circuit. Similar to Oscillogram 1 except for the time of closing the switch.

e = 60-cycle applied emf. 115 volts, effective.
i = current variation.
L = 0 092 henry. R = 3.5 ohms.
λ = 225°, approximately.

Having the voltage, current, and power variations referred to the same time axis enables one to examine their time phase relations readily.

In the oscillograms shown herein the magnitudes of the voltage and current are given as effective "steady-state" values, and the magnitude of the power is given in average "steady-state" values. Since the galvanometers have approximately straight-line calibration curves, approximate scales may be constructed for each of the oscillograms. The placement of the galvanometers in the circuit is shown in Fig. 4.

Very little transient disturbance is indicated on Oscillogram 3. The ratio of ωL to R is very low, with the result that the behavior is very much like that of an ideal R circuit. The current and power rise quickly at $t = 0$ and almost immediately assume their steady-state variations.

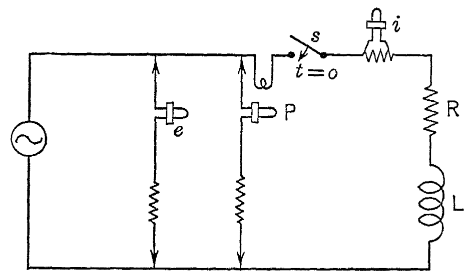


FIG. 4.—Placement of the oscillograph galvanometers in the circuit when simultaneous variations of voltage, current, and power are being recorded.

The effect of increasing the ratio of ωL to R is shown in Oscillogram 4. A greater length of time is required for the current to reach its steady state than was the case in Oscillogram 3. However, steady-state values are reached after approximately two cycles, and during this period the power fluctuates in the anticipated manner.

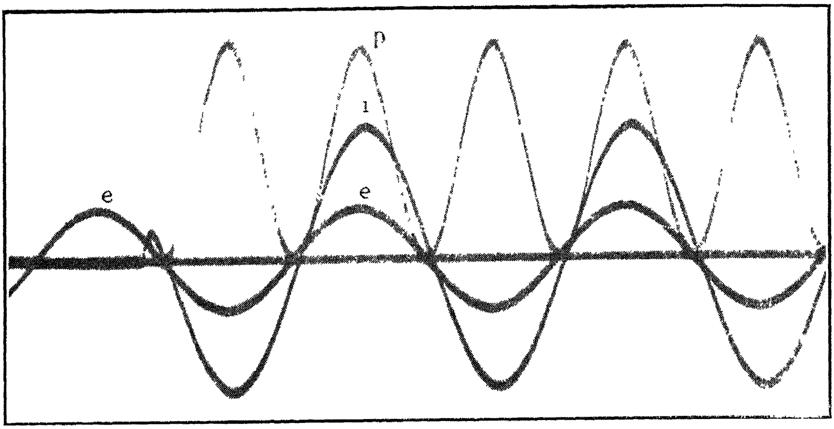
The power variation in an RL circuit having a very large ratio of ωL to R is shown in Oscillogram 5. Inasmuch as the switch is closed near $e = 0$, the fluctuations are very pronounced and persist for a relatively long period of time. The large transient component of current $(E_m/Z) \sin (\lambda - \theta) \epsilon^{-\frac{Rt}{L}}$, is responsible for the unsymmetrical variation of the power.

The expression for instantaneous power will show that, for any ratio of ωL to R, the transient disturbance is dependent upon the time at which the switch is closed:

$$p = \frac{E_m^2}{Z} \sin (\omega t + \lambda) \left[\sin (\omega t + \lambda - \theta) - \sin (\lambda - \theta) \epsilon^{-\frac{Rt}{L}} \right] \quad (22)$$

For $(\lambda - \theta) = 0$ or for $(\lambda - \theta) = \pi$ no transient term is present and the power variation immediately assumes its steady-state variation, namely:

$$p_{ss} = \frac{E_m^2}{2Z} \left[\cos \theta - \cos 2\left(\omega t + \lambda - \frac{\theta}{2} \right) \right] \quad (23)$$



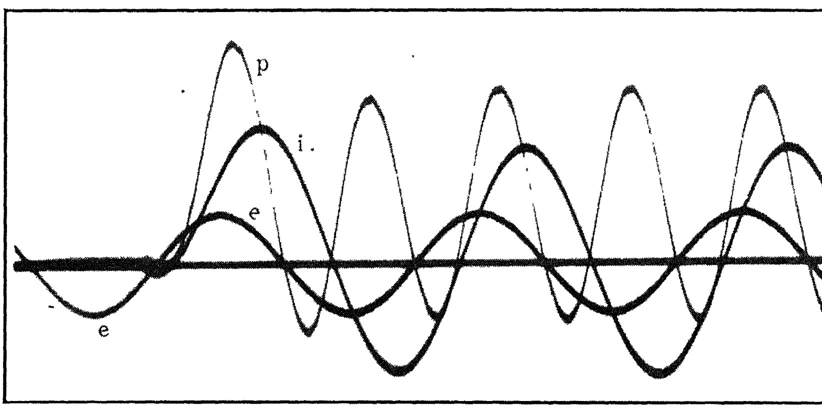
OSCILLOGRAM 3.

Voltage, current, and power in an RL circuit when the ratio of ωL to R is very low. Note that the power variation is a double frequency function and that it remains positive at all times. The product of a negative current and voltage results in positive power the same as for positive values of these quantities.

e = 60-cycle applied emf.	E (eff.) = 115 volts.
i = current variation.	I (eff.) = 3 amperes.
p = power variation.	P (avg.) = 340 watts.

Power factor of the circuit = 0.985.

λ = 160°, approximately.



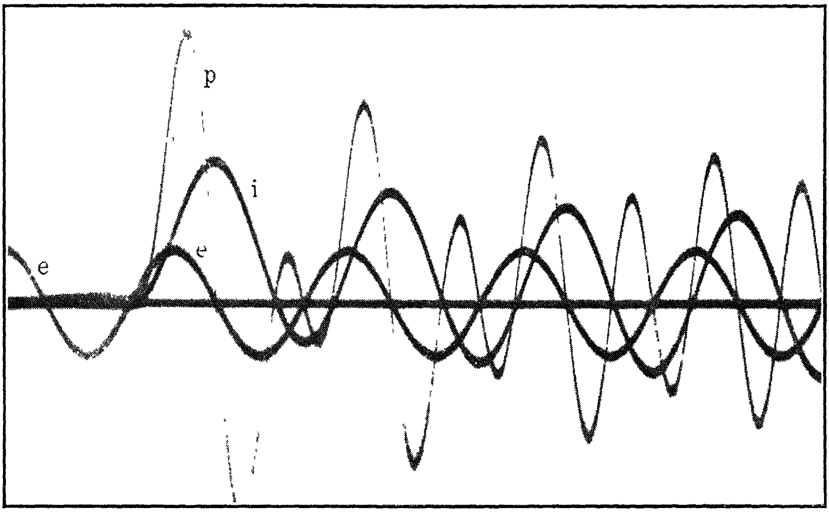
OSCILLOGRAM 4.

Similar to Oscillogram 3 except for a larger ratio of ωL to R. Note that inductance in the circuit causes the power to take on alternate positive and negative values indicating a recurring interchange of energy between the load and the source. The net power is of course still positive.

e = 60-cycle applied emf.	E (eff.) = 115 volts.
i = current variation.	I (eff.) = 2.65 amp.
p = power variation.	P (avg.) = 150 watts.

Power factor of the circuit = 0.49.

λ = − 15°, approximately.



OSCILLOGRAM 5.

Voltage, current, and power in the RL circuit wherein the ratio of ωL to R is very large. The switch is closed at approximately the time that produces maximum transient disturbance. Note that the steady-state variation of the power wave appears to be symmetrical about the zero axis, thus indicating almost complete return to the source of all energy delivered to the load.

e = 60-cycle applied emf.	E (eff.) = 115 volts.
i = current variation.	I (eff.) = 3.2 amp.
p = power variation.	P (avg.) = 37 watts.

Power factor of the circuit = 0.10.

$\lambda = 0°$, approximately.

THE RC CIRCUIT

Physical Consideration.—Many of the individual branches of filter and amplifier circuits have RC circuit characteristics. Their self-inductance is negligible as compared with their resistance and capacitance. Furthermore, with the increasing use of capacitors for power factor correction and a revival of the capacitor type of single-phase induction motor, RC circuits are becoming more common in the field of power engineering than was previously the case.

Fig. 5 is a schematic diagram of the circuit to which an alternating potential is to be applied. Let it be assumed that the condenser is uncharged prior to the time of closing the switch. For dynamic equilibrium the resistive counter-voltage and the capacitive counter-voltage must balance the applied emf. at every instant of time.

$$Ri + \frac{q}{C} = E_m \sin(\omega t + \lambda) \qquad (24)$$

where q is the instantaneous magnitude of the condenser charge. The other symbols have the same significance as they had in the previous discussion.

The circuit is, hypothetically, void of self-inductance and is therefore of the impulsive type. The current is a discontinuous function of time at the instant of closing the switch because, in general, the current is not equal to zero at that instant. Closing the switch at any time, other than $e = 0$, suddenly applies a driving emf to the circuit which may have any value between $-E_m$ and $+E_m$. Under the assumption made, the condenser is originally uncharged and hence at $t = 0$ the current instantly acquires its e/R value although just prior to this time the current was at zero value. As a result of the current flow, the condenser acquires an electric charge after a lapse of any finite interval of time. Having acquired a charge, the condenser exhibits its capacitive counter-voltage, q/C, and becomes an active member of the circuit.

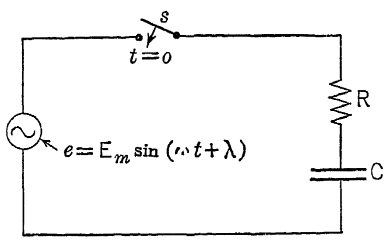


FIG. 5.—The RC circuit to which a sinusoidal voltage is applied at $t = 0$

Let the expression for the steady-state component of current again be written as follows:

$$i_s = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) \tag{25}$$

where $Z = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2}$, the steady-state impedance of the RC circuit; and $\theta = \tan^{-1} \frac{-1}{\omega CR}$, in the RC circuit. As such, θ may again be defined as the angle of lag of i_s with respect to the applied voltage. It is plain that with this definition of θ, $\frac{1}{\omega C}$ must be treated as a negative quantity and that θ will always be a negative angle of lag, that is, an angle of lead in RC circuits. The more basic reason for attaching the minus sign to $\frac{1}{\omega C}$ is that it is a circuit reactance which acts in direct opposition to ωL and, with the customary convention of signs, ωL is treated as a positive quantity.

The steady-state condenser charge may be written in the form:

$$q_s = \int i_s dt + c_1$$

$$= -\frac{E_m}{\omega Z} \cos (\omega t + \lambda - \theta) + c_1 \tag{26}$$

Under steady-state conditions:

$$Ri_s + \frac{q_s}{C} = E_m \sin (\omega t + \lambda)$$

It follows that c_1 of equation (26) must be equal to zero, and physically, that must be the case if q_s is to be a symmetrical variation about its zero axis. Since the actual condenser charge is equal to zero at $t = 0$, a transient component of condenser charge must be present at $t = 0$ which is equal in magnitude but opposite in sign to q_s at $t = 0$. A transient component will exist during the period required for the actual condenser to adjust itself to its steady-state value, q_s.

Therefore:

$$q = q_s + q_t \tag{27}$$

and

$$i = i_s + i_t \tag{28}$$

In order for dynamic equilibrium to be maintained throughout the transient period:

$$Ri_s + \frac{q_s}{C} + Ri_t + \frac{q_t}{C} = E_m \sin (\omega t + \lambda) \tag{29}$$

Recognizing that:

$$Ri_s + \frac{q_s}{C} = E_m \sin (\omega t + \lambda) \tag{30}$$

it follows that:

$$Ri_t + \frac{q_t}{C} = 0 \tag{31}$$

from which:

$$i_t = A\epsilon^{-\frac{t}{RC}} \tag{32}$$

Since

$$i = \frac{E_m}{R} \sin \lambda \qquad \text{at } t = 0$$

and

$$i = i_s + i_t$$

an evaluation of A may be effected as follows:

$$\frac{E_m}{R}\sin\lambda = \frac{E_m}{Z}\sin(\lambda - \theta) + A \tag{33}$$

$$A = \frac{E_m}{R}\sin\lambda - \frac{E_m}{Z}\sin(\lambda - \theta) \tag{34}$$

Arranged in a somewhat more convenient form:

$$A = \frac{E_m}{R}\sin\theta\cos(\lambda - \theta) \tag{35}$$

The complete expression for current becomes:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + \frac{E_m}{R}\sin\theta\cos(\lambda - \theta)\,\epsilon^{\frac{-t}{RC}} \tag{36}$$

The substitution of the right-hand member of equation (36) for i and the substitution of $\int_0^t i dt$ for q in equation (24) will show that the general condition for dynamic equilibrium is satisfied. Equation (36) is, therefore, a general expression for the current and is of much value in analyzing the behavior of the RC circuit to which a sinusoidal voltage is applied. There are two singular conditions under which the steady-state counter-voltages do balance the applied emf. The transient term is equal to zero when $\theta = 0$, and when $(\lambda - \theta) = \pi/2, 3\pi/2$, etc. The first condition is that of a purely resistive circuit. The second is that of closing the switch at the time of maximum i_s and hence at the time of zero q_s. These are the well-known conditions under which the current instantly acquires its steady-state value. With the exceptions noted above, it is evident that a transient component of current and a transient component of condenser charge must exist during the initial period, that is, until the actual current in the circuit has reached its steady-state variation.

Before proceeding, it may be well to consider the effect of an initial condenser charge, Q_0. The polarity of this charge may be either positive or negative when considered around the series loop. There are different methods of taking the effect of Q_0 into account, but a simple procedure is to include the Q_0/C voltage in the general expression for dynamic equilibrium and define q as $\int_0^t i dt$ in that expression. Considering Q_0/C as a driving voltage:

$$Ri + \frac{q}{C} = E_m\sin(\omega t + \lambda) + \frac{Q_0}{C}$$

Evaluation of A will now yield:

$$A = \frac{E_m}{R} \sin \theta \cos (\lambda - \theta) + \frac{Q_0}{RC}$$

and:

$$i_t = \left[\frac{E_m}{R} \sin \theta \cos (\lambda - \theta) + \frac{Q_0}{RC} \right] \epsilon^{-\frac{t}{RC}} \tag{37}$$

The current, $\frac{Q_0}{RC} \epsilon^{-\frac{t}{RC}}$, is the normal condenser discharge current due to the initial charge, Q_0. It is a positive current if Q_0/C is positive, that is, if Q_0/C acting as a driving voltage sends current around the series loop in the positive direction.

A third condition under which the transient current may be zero appears when the initial condenser charge is of such a value that the two terms within the bracket of equation (37) are equal in magnitude and opposite in sign. In this case the Q_0/C voltage cancels the other discrepancy voltage, and under these conditions the Ri_s and q_s/C counter-voltages are capable of balancing the applied emf.

With an initial condenser charge, the general expression for current is:

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + \left[\frac{E_m}{R} \sin \theta \cos (\lambda - \theta) + \frac{Q_0}{RC} \right] \epsilon^{-\frac{t}{RC}} \tag{38}$$

An examination of this expression reveals the general nature of the current variation with respect to time. The first member of the right-hand side of the equation is the well-known steady-state component. It will be remembered that θ, as defined, is always negative in the RC circuit and is actually the angle of lead of the steady-state component of current with respect to the voltage. The second member is the transient component of the current. Its initial magnitude depends upon the several factors which combine to form the bracket coefficient of $\epsilon^{-\frac{t}{RC}}$. Starting at this particular initial value, the transient component decreases exponentially. The magnitude of the product RC governs the rate of subsidence. In ordinary RC circuits the transient term disappears rather rapidly. The plotted solution of equation (38) is shown in Fig. 6, for a particular set of circuit parameters and a particular time of closing the switch. For the case illustrated, $\lambda = -30°$ and $\theta = -60°$. A wide variety of current variations near $t = 0$ may be obtained by the choice of other conditions.

Mathematical Analysis.—The complete expression for the current might have been obtained in a much more direct manner and with

little consideration of the physical phenomena involved except a knowledge of the boundary conditions. i and q in equation (29) need not have been broken up into their steady-state and transient components. Considering these quantities in their entirety and differentiating equation (24) results in:

$$R\frac{di}{dt} + \frac{i}{C} = \omega E_m \cos(\omega t + \lambda) \tag{39}$$

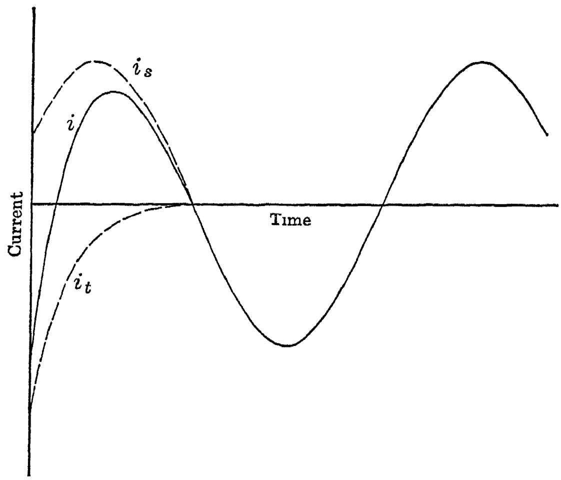


FIG. 6.—Graph of equation (38), illustrating the transient behavior of the RC circuit

$$e = 162.5 \sin\left(377t - \frac{\pi}{12}\right) \quad \text{(not shown)}$$
$$i = 1.48 \sin(377t + 30°) - 2.21\epsilon^{-652t}$$
$$R = 55 \text{ ohms.} \quad C = 27.9\mu\text{f.} \quad Q_0 = 0.$$

The general procedure connected with the mathematical solution is similar to that which was given for equation (9). It will, of course, be recognized that the boundary conditions must be imposed on equation (24) and not on the differentiated form given in (39).

The Operational Solution.—The general condition for dynamic equilibrium in the RC circuit to which sinusoidal voltage is suddenly applied may be written in the following manner:

$$Ri + \frac{i}{pC} = E_m \sin(\omega t + \lambda) + \frac{Q_0}{C} \tag{40}$$

Solving for i:

$$i = \frac{E_m \sin(\omega t + \lambda)}{R + \frac{1}{pC}} + \frac{1}{R + \frac{1}{pC}}\frac{Q_0}{C} \tag{41}$$

$$= \frac{E_m}{R} \cdot \frac{p}{p + \frac{1}{RC}} \sin (\omega t + \lambda) + \frac{Q_0}{RC} \cdot \frac{p}{p + \frac{1}{RC}} 1 \qquad (42)$$

It will be noted that in one term the operation is to be performed on a function of time, and in the other, on a constant. The details connected with the operation of $p/(p + a)$ on $\sin (\omega t + \lambda)$ may be found on page 290, where it is shown that:

$$\frac{p}{p + a} \sin (\omega t + \lambda) = \frac{\omega}{\sqrt{a^2 + \omega^2}} \sin (\omega t + \lambda - \theta) + \sin \theta \cos (\lambda - \theta) \epsilon^{-at}$$

where $\theta = \tan^{-1} \frac{-a}{\omega}$.

The meaning of $p/(p + a)$ operating on a constant has been shown to be equal to ϵ^{-at}.

For the case under consideration:

$$a = \frac{1}{RC}$$

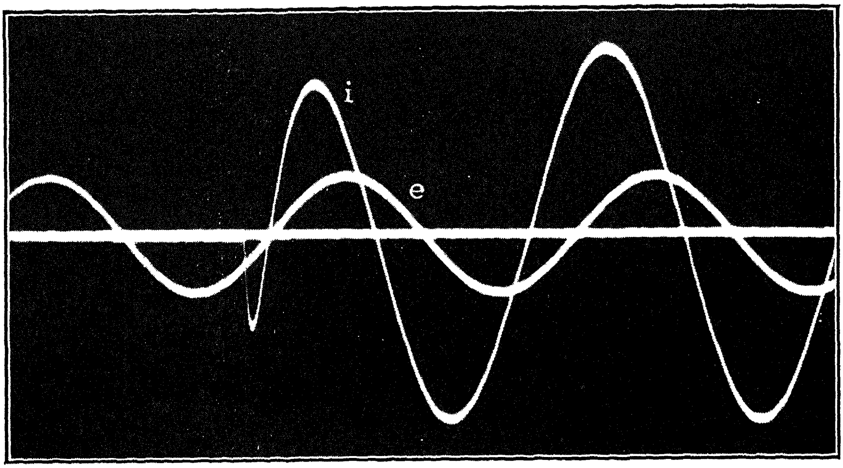
$$\theta = \tan^{-1} \frac{-1}{\omega CR}, \text{ the p.f. angle of the circuit.}$$

Performing the operations indicated in equation (42):

$$i = \frac{E_m}{R} \cdot \frac{\omega}{\sqrt{\frac{1}{R^2C^2} + \omega^2}} \sin (\omega t + \lambda - \theta) + \left[\frac{E_m}{R} \sin \theta \cos (\lambda - \theta) + \frac{Q_0}{RC} \right] \epsilon^{-\frac{t}{RC}}$$

$$= \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + \left[\frac{E_m}{R} \sin \theta \cos (\lambda - \theta) + \frac{Q_0}{RC} \right] \epsilon^{-\frac{t}{RC}} \qquad (43)$$

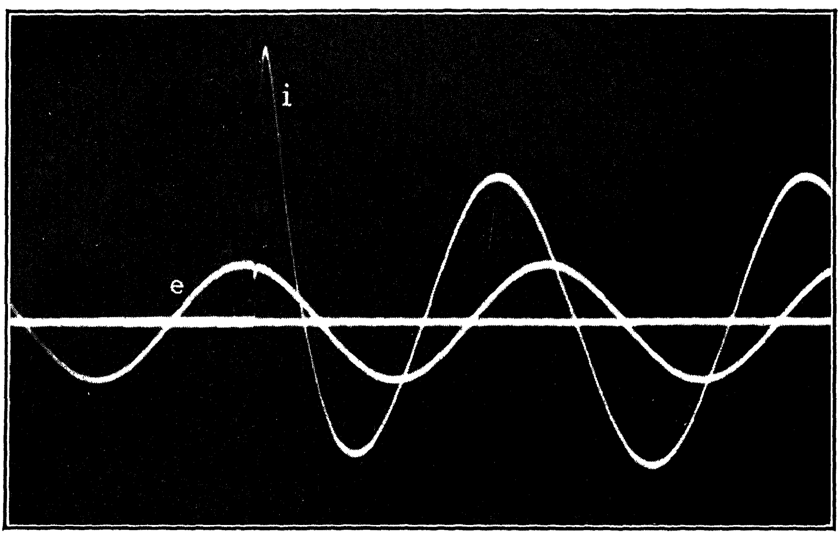
Oscillographic Verification.—Certain problems present themselves when an attempt is made to study the *RC* circuit experimentally. The wave form of the applied emf must be very nearly sinusoidal if the experimentally determined results are to be compared with those that have been calculated upon the basis of sinusoidal voltage variation. A sine wave alternator is required, and in the ordinary laboratory these machines are of relatively low kilovolt-ampere capacity with correspondingly high leakage reactance. The self-inductance of the armature winding may therefore be an appreciable parameter of the completed circuit. Oscillogram 6 illustrates the effect. It will be noted from the legend that the oscillogram was taken under conditions which were very similar to those for which the graphs in Fig. 6 were calculated. The current does not instantly acquire its e/R value and never does



OSCILLOGRAM 6.

Transient current in the RC circuit.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation: I (eff.) = 1.05 amp.
$C = 28\mu$f. $R = 55$ ohms.
$\lambda = -30°$. approximately. $Q_0 = 0$.



OSCILLOGRAM 7.

Illustrating the effect of Q_0 in the RC circuit.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. I (eff.) = 1.05 amp.
$C = 28\mu$f. $R = 55$ ohms.
$\lambda = 105°$, approximately. $Q_0/C = +90$ volts.

attain as large negative values as would be present in the pure *RC* circuit. But the general nature of the *RC* variation is in evidence. The negative transient component of the current is of sufficient magnitude to affect the actual current during the early period of its variation and give it the general trend of the current in an ideal *RC* circuit. It is interesting to observe the rapid rate at which the transient component dies out and the manner in which the actual current of the circuit acquires its steady-state variation. The inductive reactance of the circuit is negligible as compared with the capacitive reactance as far as the circuit's steady-state behavior is concerned.

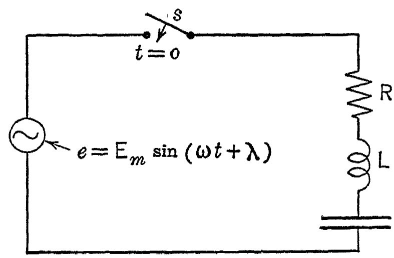


FIG. 7.—The *RLC* circuit to which a sinusoidal voltage is applied at $t = 0$.

The effect of initial condenser charge upon current flow is shown quite clearly in Oscillogram 7. Prior to closing the switch the condenser possesses a positive Q_0 such that Q_0/C is equal to $+$ 90 volts. For the purpose of oscillographic analysis, a voltage is considered positive if it produces a current which is positive, that is, upward from the ground line. For the case shown in Oscillogram 7 the transient component of the current is large and positive since both $\left[\frac{E_m}{R}\sin\theta\cos(\lambda-\theta)\right]$ and $\left[\frac{Q_0}{RC}\right]$ are positive. The value of the steady-state component near $t = 0$ is small.

Transient Power.—The expression for the instantaneous power delivered to the *RC* circuit is at once obtainable. Multiplying the expression for current as given by equation (43), by the expression for the instantaneous voltage, results in the following:

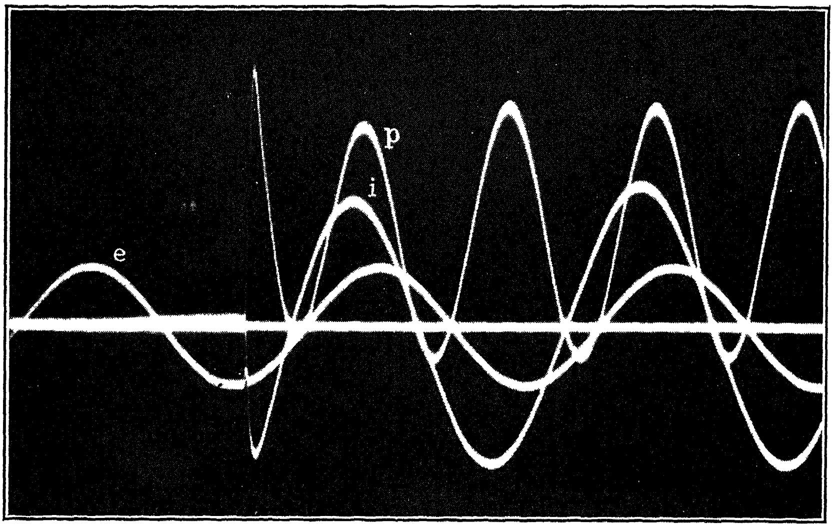
$$p = \frac{E_m^2}{2Z}\left[\cos\theta - \cos\{2(\omega t + \lambda) - \theta\}\right]$$
$$+ \left[\frac{E_m^2}{R}\sin\theta\cos(\lambda - \theta) + \frac{E_m Q_0}{RC}\right]\sin(\omega t + \lambda)\epsilon^{-\frac{t}{RC}} \qquad (44)$$

Equation (44) is of some value in predicting the general nature of power variations, but in particular cases, the plotted solutions may be obtained much more easily by means of oscillograms.

Oscillogram 8 illustrates the voltage, current, and power variations in the RC circuit in a particular case. The case illustrated is that of:

$$\theta = \tan^{-1} 1.0 = 45°$$

The impulsiveness of the circuit is indicated by the sudden change in current and power at $t = 0$.



OSCILLOGRAM 8

Voltage, current, and power variations in the RC circuit.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. I (eff.) = 2 70 amp.
p = power variation. P (avg.) = 220 watts.

Power factor of circuit = 0.707. $\theta = -45°$.
$\lambda = -80°$, approximately. $Q_0 = 0$.

THE *RLC* CIRCUIT

Physical Considerations.—The natural behavior of this circuit with continuous voltage excitation was considered rather completely in Chapter II, and it will be remembered that a wide variety of phenomena was exhibited. However, no steady-state component of current persisted. With the appearance of a steady-state component of current the study acquires additional aspects. The general nature of the circuit's behavior with sinusoidal applied voltage will be more or less obvious in the light of previous considerations, but it remains to work out the details and in so doing become more fully acquainted with this circuit.

Three component counter-voltages are active in balancing the applied emf.

$$Ri + L\frac{di}{dt} + \frac{q}{C} = E_m \sin(\omega t + \lambda) \tag{45}$$

The steady-state component of current is known to be:

$$i_s = \frac{E_m}{Z} \sin(\omega t + \lambda - \theta) \tag{46}$$

where

$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$, the steady-state impedance of the circuit.

$\theta = \tan^{-1} \dfrac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$, the angle of lag or lead of the steady-state component of current with respect to the applied emf.

For the sake of analysis, i in equation (45) is separated into its "transient" and "steady-state" components, namely, i_t and i_s

$$i = i_s + i_t \tag{47}$$

Differentiating equation (45) gives:

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = E_m\omega \cos(\omega t + \lambda) \tag{48}$$

Since:

$$R\frac{di_s}{dt} + L\frac{d^2i_s}{dt^2} + \frac{i_s}{C} = E_m\omega \cos(\omega t + \lambda) \tag{49}$$

it follows that:

$$R\frac{di_t}{dt} + L\frac{d^2i_t}{dt^2} + \frac{i_t}{C} = 0 \tag{50}$$

The details connected with the solution of equation (50) are given in Chapter II, and result in:

$$i_t = A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \tag{51}$$

where

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a + b \tag{52}$$

and

$$\alpha_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a - b \tag{53}$$

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \tag{54}$$

The boundary conditions are assumed to be:

$$i = 0 \quad \text{at} \quad t = 0$$

and

$$q = Q_0 \quad \text{at} \quad t = 0$$

The following relationships between A_1 and A_2 are obtained by imposing the boundary conditions upon equations (45) and (54):

$$\frac{E_m\omega L}{Z}\cos(\lambda - \theta) + (A_1\alpha_1 + A_2\alpha_2)L + \frac{Q_0}{C} = E_m \sin\lambda \tag{55}$$

$$A_1 + A_2 = -\frac{E_m}{Z}\sin(\lambda - \theta) \tag{56}$$

The values of A_1 and A_2 follow directly from the relations stated above.

$$A_1 = \frac{E_d'}{2bL} - \frac{E_m}{2Z}\sin(\lambda - \theta) \tag{57}$$

$$A_2 = -\frac{E_d'}{2bL} - \frac{E_m}{2Z}\sin(\lambda - \theta) \tag{58}$$

where, for the sake of simplicity in writing,

$$E_d' = \left[E_m \sin\lambda - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - \frac{Q_0}{C} - \frac{E_m R}{2Z}\sin(\lambda - \theta)\right] \tag{59}$$

The complete expression for the current may now be written:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + A_1\epsilon^{\alpha_1 t} + A_2\epsilon^{\alpha_2 t} \tag{60}$$

$$= \frac{E_m}{Z}\sin(\omega t + \lambda - \theta)$$

$$+ \epsilon^{-at}\left\{\frac{E_d'}{bL}\frac{(\epsilon^{bt} - \epsilon^{-bt})}{2} - \frac{E_m}{Z}\sin(\lambda - \theta)\frac{(\epsilon^{bt} + \epsilon^{-bt})}{2}\right\} \tag{61}$$

The above expression will, of course, reveal those facts or conditions which were employed in its development. The two components of cur-

rent are distinctly represented. At $t = 0$, the transient component is equal in magnitude and opposite in sign to that of the steady-state component. After a reasonable lapse of time the current is, sensibly, expressed by the steady-state component. Several other significant features are clearly shown by equation (61). The transient component of current consists of two terms. One is exactly similar in form to that found for constant voltage excitation. It will be observed that the driving voltage for this term is E'_d. The second transient term is that part which is equal in magnitude to i_s at $t = 0$ but opposite in sign. Since the first of the two terms is zero at $t = 0$, the second represents the transient component of current at $t = 0$. The entire transient component is damped out by the factor, $\epsilon^{-\frac{Rt}{2L}}$. The fact that b may be either a real quantity, an imaginary quantity, or equal to zero gives equation (61) three distinct forms.

Graphs and Oscillograms for Case I

When

$$\frac{R^2}{4L^2} > \frac{1}{LC}:$$

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + \epsilon^{-at}\left\{\frac{E_d'}{bL} \sinh bt - \frac{E_m}{Z} \sin (\lambda - \theta) \cosh bt\right\} \quad (62)$$

The general trends of the two transient terms are evident. One is an exponentially damped hyperbolic sine similar to the current variation in this type of circuit under the condition of constant voltage excitation. The other is that of an exponentially damped hyperbolic cosine. The graphs of i_{t1} and i_{t2} together with that of i_s are shown in Fig. 8 for a particular case.

$$i_{t1} = \epsilon^{-at}\left[\frac{E_d'}{bL} \sinh bt\right] \quad (63)$$

$$i_{t2} = -\ \epsilon^{-at}\left[\frac{E_m}{Z} \sin (\lambda - \theta) \cosh bt\right] \quad (64)$$

In Fig. 8 the plotted solution of i, the actual current in the circuit, has been obtained by adding the graphs of i_s, i_{t1}, and i_{t2}. A comparison of the plotted solution may be made with the experimentally determined results shown in Oscillogram 9. It will be observed that the circuit parameters and boundary conditions are the same for Oscillogram 9 as for the plotted solution shown in Fig. 8.

Graphs and Oscillograms for Case II

$$\frac{R^2}{4L^2} < \frac{1}{LC}$$

Since, in this case, b is an imaginary quantity a change in nomenclature is desirable. As in Chapter II, it is convenient at this point to let:

$$b = j\beta \tag{65}$$

where

$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{66}$$

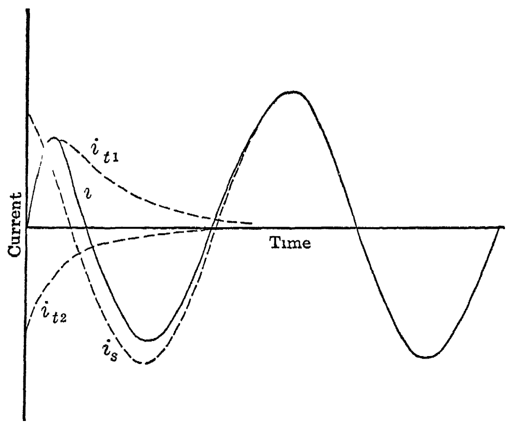


Fig. 8.—Plotted solution of equation (62) for a particular set of parameters, illustrating the transient current in the RLC circuit when $\frac{R^2}{4L^2} > \frac{1}{LC}$ The component currents which combine to form the resultant current are indicated by the dotted graphs.

$e = 162.5 \sin (377t + 110°)$ (not shown).
$R = 110$ ohms. $L = 0.094$ henry $C = 42\mu$f. $Q_0 = 0$.
$i = 1.43 \sin (377t + 124° 6') + \epsilon^{-585t} [4.11 \sinh 300t - 1.19 \cosh 300t]$.

Equation (61) reduces at once to the following form:

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + \epsilon^{-at}\left\{\frac{E_d'}{\beta L} \sin \beta t - \frac{E_m}{Z} \sin (\lambda - \theta) \cos \beta t\right\} \tag{67}$$

The only difference between equation (67) and equation (62) is in the two transient terms. In the present case these two terms acquire sinusoidal and cosinusoidal variations. They may be combined so that:

$$i_t = \epsilon^{-at}\{I_t' \sin (\beta t - \sigma)\} \tag{68}$$

where

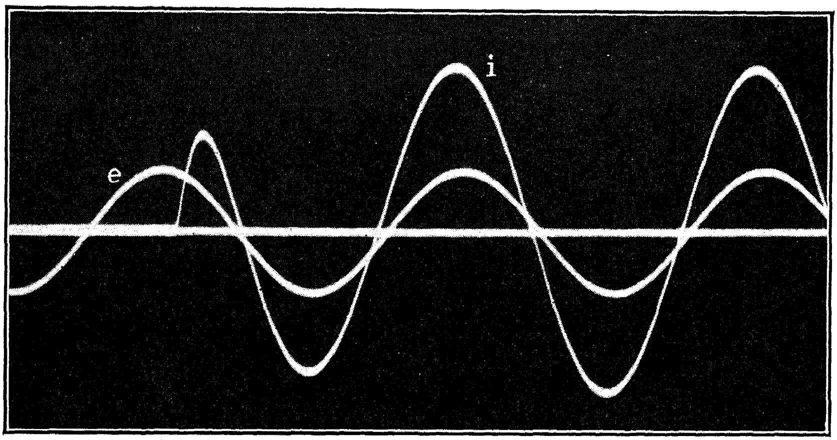
$$I_t' = \sqrt{\left[\frac{E_d'}{\beta L}\right]^2 + \left[\frac{E_m}{Z}\sin(\lambda - \theta)\right]^2}$$

and

$$\sigma = \tan^{-1}\frac{E_m\beta L \sin(\lambda - \theta)}{E'_d Z}$$

Using the above nomenclature, equation (67) may be written:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + \epsilon^{-at} I_t' \sin(\beta t - \sigma) \tag{69}$$



OSCILLOGRAM 9.

Transient current in the *RLC* circuit. $\frac{R^2}{4L^2} > \frac{1}{LC}$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. I (eff.) = 1.0 amp.

$R = 110$ ohms. $L = 0.094$ henry. $C = 42\mu$f.
$\lambda = 110°$, approximately. $Q_0 = 0$.

Equation (69) is in a convenient form for making plotted solutions. The graph of the transient component is an exponentially damped sine wave, starting its variation at $t = 0$ with a value equal to $[-I'_t \sin \sigma]$.

The plotted solution of equation (69) for a particular set of circuit parameters and boundary conditions is shown in Fig. 9. The circuit parameters and boundary conditions have been chosen so that they agree with those of Oscillogram 10. A comparison may therefore be made between the calculated values and those determined experimentally.

In certain cases the transient component of the current may be several times as large as the steady-state term. The graphical solution

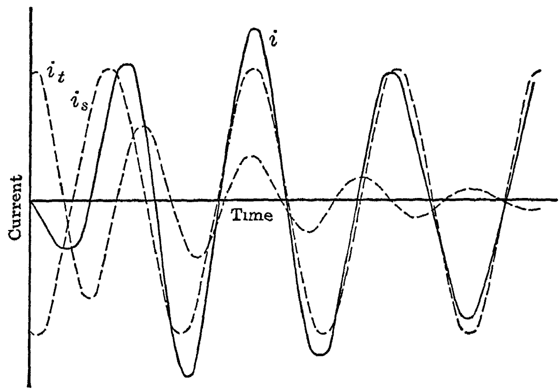


FIG. 9.—Plotted solution of equation (69) for a particular set of parameters illustrating the transient current in the RLC circuit when $\frac{R^2}{4L^2} < \frac{1}{LC}$. The component currents which combine to form the resultant current are indicated by the dotted graphs.

$e = 162\,5 \sin (377t + 185°)$ (not shown).
$R = 8.5$ ohms. $L = 0.094$ henry. $C = 42\mu$f. $Q_0 = 0$.
$i = 5.62 \sin (377t + 252° 55') + 5.64\epsilon^{-45\,2t} \sin (501t + 72° 5')$.

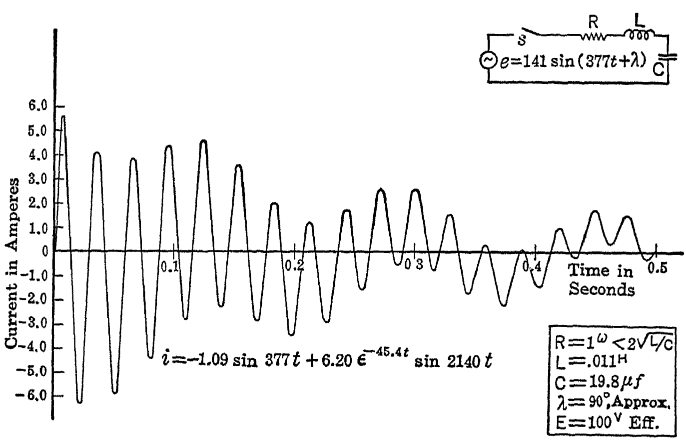
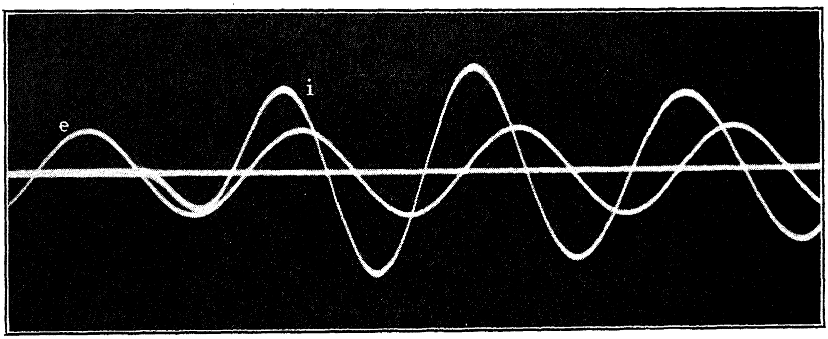


FIG. 10.—The graph of the resultant current shown is composed of a relatively high-frequency transient component and a 60-cycle steady-state component. The maximum magnitude of the transient component is approximately six times that of the steady-state component near $t = 0$.

of equation (69) shown in Fig. 10 illustrates a case wherein I_t' is approximately six times as large as the maximum value of the steady-state

term. The angular velocity of the transient component is 2140 radians per second as compared with 377 radians per second, the angular velocity of the steady-state component.

The frequency of the oscillating transient component is independent of the frequency of the steady-state component. It may be higher than, equal to, or lower than the steady-state frequency since its period is governed entirely by the relative values of R, L, and C.



OSCILLOGRAM 10.

Transient current in the *RLC* circuit. $\dfrac{R^2}{4L^2} < \dfrac{1}{LC}$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. I (eff.) = 3.85 amp.

$R = 8.5$ ohms. $L = 0.094$ henry. $C = 42\mu$f.
$\lambda = 180°$, approximately. $Q_0 = 0$.

THE TRANSITION FROM CASE I TO CASE II. CASE III.

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

The boundary line between Cases I and II is generally referred to as the critical case. When the above relationship between the three circuit parameters exists β is equal to zero and the expression for current may be obtained directly from equation (67). Recognizing that:

$$\lim_{\beta \to 0} \left[\frac{\sin \beta t}{\beta} \right] = t,$$

$$i = \frac{E_m}{Z} \sin(\omega t + \lambda - \theta) + \epsilon^{-\alpha t} \left\{ \frac{E_d'}{L} t - \frac{E_m}{Z} \sin(\lambda - \theta) \right\} \tag{70}$$

In plotted form, the component parts of equation (70) resemble, somewhat, the corresponding parts of equation (62) which are shown in Fig. 8.

Mathematical Analysis.—For convenience, equation (45) is rewritten at this point.

$$Ri + L\frac{di}{dt} + \frac{q}{C} = E_m \sin(\omega t + \lambda) \tag{45}$$

It is an important form of differential equation because it frequently occurs in physical investigations. There are several conventional methods of solution, one of which will be outlined. Differentiating and rearranging equation (45) it may be made to take the following form:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{\omega E_m}{L}\cos(\omega t + \lambda) \tag{71}$$

If it be assumed that R, L, C, ω, λ, and E_m are constant, equation (71) admits of relatively simple solution by the symbolic operator method. A general solution of second order differential equations by the symbolic operator method may be found on page 272. A brief review, as applied to this particular problem, is given for those who are not familiar with the details.

If the proper interpretation of $(D - \alpha_1)$ and $(D - \alpha_2)$ is made, equation (71) may be written in the following symbolic form:

$$(D - \alpha_1)(D - \alpha_2)i = \frac{\omega E_m}{L}\cos(\omega t + \lambda) \tag{72}$$

where α_1 and α_2 are constants, yet to be determined, and

$$D = \frac{d}{dt}$$

$$(D - \alpha_1)i = \frac{di}{dt} - \alpha_1 i$$

$$(D - \alpha_2)i = \frac{di}{dt} - \alpha_2 i$$

Therefore:

$$(D - \alpha_1)\left(\frac{di}{dt} - \alpha_2 i\right) = \frac{\omega E_m}{L}\cos(\omega t + \lambda) \tag{73}$$

Performing the operations indicated above, the equation takes the following form:

$$\frac{d^2i}{dt^2} - (\alpha_1 + \alpha_2)\frac{di}{dt} + \alpha_1\alpha_2 i = \frac{\omega E_m}{L}\cos(\omega t + \lambda) \tag{74}$$

The order of operation is immaterial. $(D - \alpha_2)$ operating on $\left(\frac{di}{dt} - \alpha_1 i\right)$

would have given the same result. A comparison of equations (71) and (74) shows that the foregoing use of the symbolic operators, $(D - \alpha_1)$ and $(D - \alpha_2)$, is valid. It is also evident that α_1 and α_2 are roots of $\left[D^2 + \frac{R}{L}D + \frac{1}{LC} = 0\right]$ treated as an algebraic equation because $(D - \alpha_1)$ and $(D - \alpha_2)$ treated as algebraic expressions are factors of that equation. Thus:

$$\alpha_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a + b \tag{75}$$

$$\alpha_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a - b \tag{76}$$

It remains to solve equation (72) for i. A simple procedure is to solve explicitly for i and separate the result into partial fractions before attempting to perform the operations which are indicated by the D's.

$$i = \frac{\frac{\omega E_m}{L}\cos(\omega t + \lambda)}{(D - \alpha_1)(D - \alpha_2)} \tag{77}$$

Separating into partial fractions:

$$i = \frac{\omega E_m \cos(\omega t + \lambda)}{L(\alpha_1 - \alpha_2)(D - \alpha_1)} - \frac{\omega E_m \cos(\omega t + \lambda)}{L(\alpha_1 - \alpha_2)(D - \alpha_2)} \tag{78}$$

In the above form the complete expression for i may be considered as the sum of two components, i_1 and i_2.

$$i = i_1 + i_2 \tag{79}$$

where

$$i_1 = \frac{\omega E_m \cos(\omega t + \lambda)}{L(\alpha_1 - \alpha_2)(D - \alpha_1)} \tag{80}$$

$$i_2 = -\frac{\omega E_m \cos(\omega t + \lambda)}{L(\alpha_1 - \alpha_2)(D - \alpha_2)} \tag{81}$$

Then,

$$\frac{di_1}{dt} - \alpha_1 i_1 = \frac{\omega E_m \cos(\omega t + \lambda)}{L(\alpha_1 - \alpha_2)} \tag{82}$$

$$\frac{di_2}{dt} - \alpha_2 i_2 = -\frac{\omega E_m \cos(\omega t + \lambda)}{L(\alpha_1 - \alpha_2)} \tag{83}$$

Equations (82) and (83) are similar to equation (9) in form and may be solved in a similar manner.

The general solutions of equations (82) and (83) may be written as follows:

$$i_1 = \frac{\omega E_m}{L(\alpha_1 - \alpha_2)}\left\{\frac{\omega \sin(\omega t + \lambda) - \alpha_1 \cos(\omega t + \lambda)}{\alpha_1^2 + \omega^2}\right\} + c_1 \epsilon^{\alpha_1 t} \tag{84}$$

$$i_2 = \frac{-\omega E_m}{L(\alpha_1 - \alpha_2)}\left\{\frac{\omega \sin(\omega t + \lambda) - \alpha_2 \cos(\omega t + \lambda)}{\alpha_2^2 + \omega^2}\right\} + c_2 \epsilon^{\alpha_2 t} \tag{85}$$

Adding i_1 and i_2 and making certain reductions:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + c_1 \epsilon^{\alpha_1 t} + c_2 \epsilon^{\alpha_2 t} \tag{86}$$

where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\theta = \tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

c_1 and c_2 are constants of integration

Let it be assumed that the following boundary conditions exist:

$$i = 0 \quad \text{at} \quad t = 0$$

$$q = Q_0 \quad \text{at} \quad t = 0$$

Imposing these conditions upon equation (45) yields:

$$L\left(\frac{di}{dt}\right)_{t=0} = E_m \sin\lambda - \frac{Q_0}{C} \tag{87}$$

From equation (86),

$$\left(\frac{di}{dt}\right)_{t=0} = \frac{E_m \omega}{Z}\cos(\lambda - \theta) + c_1\alpha_1 + c_2\alpha_2 \tag{88}$$

It follows that:

$$\alpha_1 c_1 + \alpha_2 c_2 = \frac{E_m}{L}\sin\lambda - \frac{E_m\omega}{Z}\cos(\lambda - \theta) - \frac{Q_0}{LC} \tag{89}$$

Equation (86) may be employed to establish:

$$c_1 + c_2 = -\frac{E_m}{Z}\sin(\lambda - \theta) \tag{90}$$

From the relations stated in (89) and (90):

$$c_1 = \frac{1}{2bL}\left[E_m \sin\lambda - \frac{E_m\omega L}{Z}\cos(\lambda-\theta) - \frac{Q_0}{C}\right] + \frac{\alpha_2 E_m}{2bZ}\sin(\lambda-\theta) \quad (91)$$

$$= \frac{1}{2bL}\left[E_m \sin\lambda - \frac{E_m R}{2Z}\sin(\lambda-\theta) - \frac{E_m\omega L}{Z}\cos(\lambda-\theta) - \frac{Q_0}{C}\right] - \left[\frac{E_m}{2Z}\sin(\lambda-\theta)\right] \quad (92)$$

and:

$$c_2 = -\frac{1}{2bL}\left[E_m \sin\lambda - \frac{E_m R}{2Z}\sin(\lambda-\theta) - \frac{E_m\omega L}{Z}\cos(\lambda-\theta) - \frac{Q_0}{C}\right] - \left[\frac{E_m}{2Z}\sin(\lambda-\theta)\right] \quad (93)$$

The complete expression for i thus becomes:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + \epsilon^{-at}\left\{\frac{E_d'}{bL}\frac{(\epsilon^{bt} - \epsilon^{-bt})}{2} - \frac{E_m}{Z}\sin(\lambda-\theta)\frac{(\epsilon^{bt} + \epsilon^{-bt})}{2}\right\} \quad (94)$$

where

$$E'_d = \left[E_m \sin\lambda - \frac{E_m R}{2Z}\sin(\lambda-\theta) - \frac{E_m\omega L}{Z}\cos(\lambda-\theta) - \frac{Q_0}{C}\right] \quad (95)$$

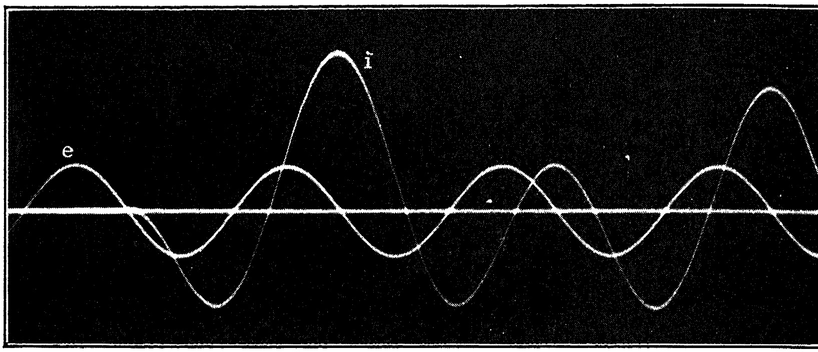
Since the foregoing analysis is essentially an operational solution the Heaviside method will not be given.

Oscillographic Verification.—Photographic records of current variation have been used to illustrate the two general forms of the expression for current in the *RLC* circuit. There are, of course, an infinite number of possible types of current variation since the number of variables is large. However, the degree of difference, within a particular case, is most striking in number two of the general cases. Here the transient component of current oscillates at a definite frequency which is defined by the parameters of the circuit. Equations (67) and (69) describe the nature of the exponentially damped oscillation that constitutes the transient component of the actual current.

Oscillogram 11 illustrates the effect of a low-frequency transient component. In this case β is less than ω and the actual current is the resultant of a 60-cycle steady-state component and a 29-cycle transient

component. The cancelling effects of the two components in certain regions is noticeable.

A singular case is shown in Oscillogram 12. Within the limits of

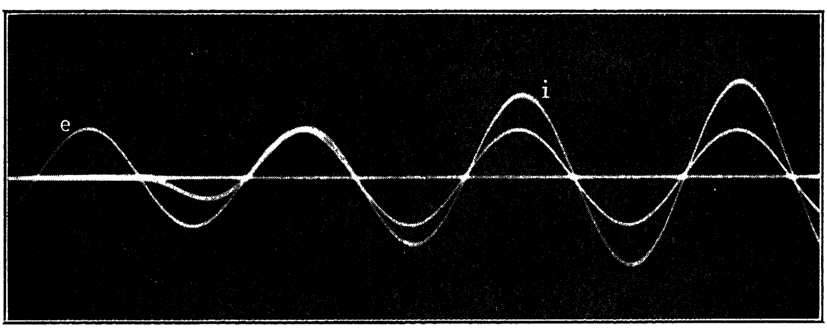


OSCILLOGRAM 11.

Illustrating the effect of a low-frequency transient component upon the resultant current in the *RLC* circuit. $\beta < \omega$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. I (eff.) = 4.2 amp.

$R = 4$ ohms. $L = 0.094$ henry. $C = 320\mu$f.
$\lambda = 180°$, approximately. $Q_0 = 0$.



OSCILLOGRAM 12.

The gradual growth of current in the *RLC* circuit when $\beta = \omega$.

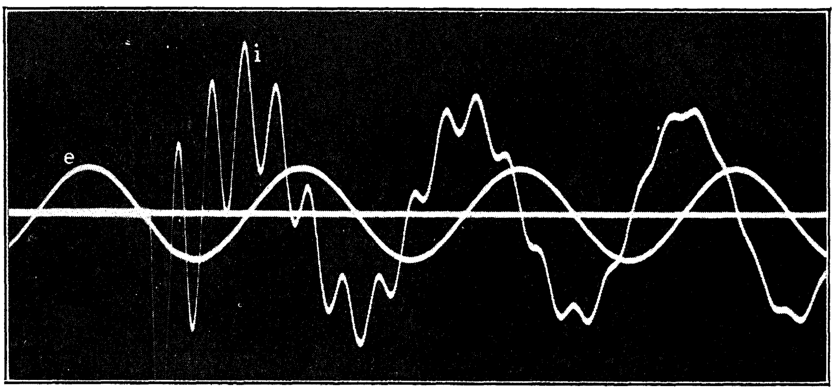
e = 60-cycle sinusoidal applied emf. E (eff.) = 44 volts.
i = current variation. I (eff.) = 4.7 amp.

$R = 9.3$ ohms. $L = 0.094$ henry. $C = 75\mu$f.
$\lambda = 180°$ approximately. $Q_0 = 0$.

oscillographic accuracy, the natural frequency of the circuit is equal to the frequency of the applied emf. Under these conditions, $\beta = \omega$. The effect of the transient component of current is clearly illustrated

in the oscillogram. Picturing the steady-state component at and near $t = 0$, the nature of the transient component may be gained by inspection. It is an exponentially damped 60-cycle variation, opposite in polarity to the steady-state component. As the transient component subsides the resultant current gradually acquires its steady-state value. The outstanding characteristic in this particular case is the smoothness with which the current grows into its steady-state variation.

The transient component of the current is most easily discerned on an oscillogram when the natural frequency of the circuit is several times that of the applied emf. Oscillogram 13 illustrates the case of a



OSCILLOGRAM 13.

Illustrating the effect of a high-frequency transient component upon the resultant current in the *RLC* circuit. $\beta > \omega$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.

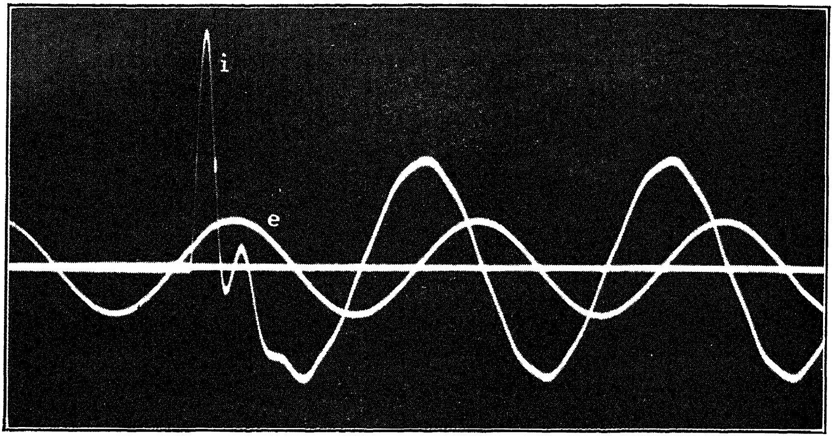
i = current variation. I (eff.) = 0.67 amp.

$R = 2$ ohms. $L = 0.012$ henry. $C = 14.9\ \mu$f.

$\lambda = 190°$, approximately. $Q_0 = 0$.

high-frequency transient component superimposed on a 60-cycle steady-state component. The 380-cycle variation, as defined by the parameters of the circuit, is very much in evidence during the early part of the transient period. The fact that the circuit is highly capacitive results in a rather irregular steady-state current.

Increasing the resistance materially shortens the time required for the current to reach its steady-state variation. The damping constant a is directly proportional to R. Oscillogram 14 was taken under practically the same conditions as 13 except for the increase in resistance. The increase in resistance does not materially affect the natural period of the circuit. It will be observed, however, that the high-frequency component is damped out in less than one cycle of the steady-state

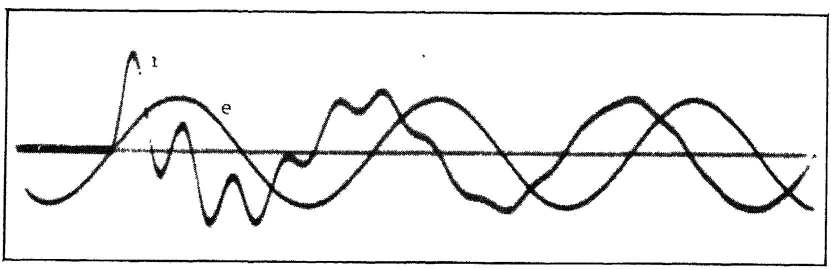


OSCILLOGRAM 14.

Compared with Oscillogram 13, the above oscillogram illustrates the effect of increasing R in the RLC circuit.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. I (eff.) = 0.63 amp.

R = 12 ohms. L = 0.012 henry. C = 14.9μf.
λ = 15°, approximately. $Q_0 = 0$.



OSCILLOGRAM 15.

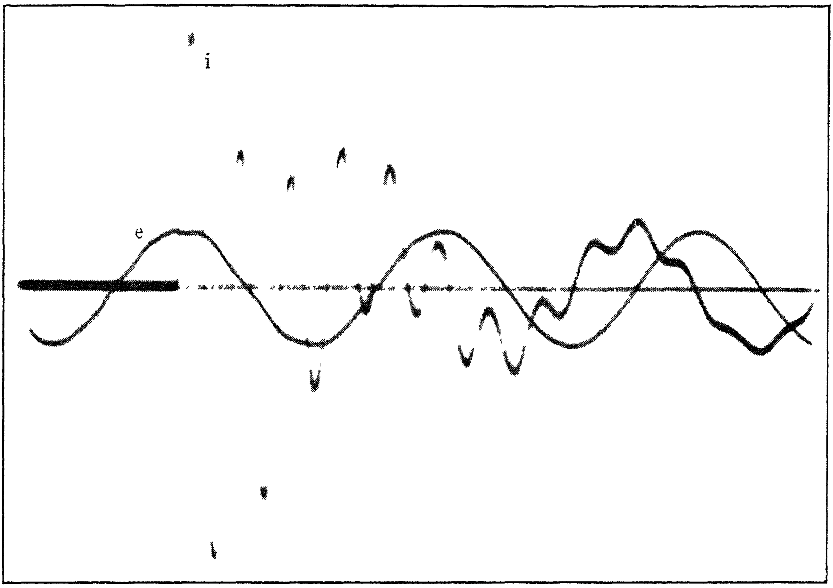
Compared with Oscillogram 16, the above oscillogram illustrates the relatively small transient effect due to closing the circuit at $e = 0$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 35 volts.
i = current variation. I (eff.) = 0.27 amp.

R = 2.0 ohms. L = 0.012 henry. C = 19.8μf.
$\lambda = 0°$. $Q_0 = 0$.

variation. The change in wave form of the steady-state current is also evident.

λ is a significant factor in determining the magnitude of E_d', and, other things being equal, the magnitude of the transient current is greatly influenced by the time of closure of the circuit. Oscillograms 15 and 16 illustrate the marked difference in the transient current between the cases of $\lambda = 0$ and $\lambda = 90°$ for a given set of circuit parameters.



OSCILLOGRAM 16.

Compared with Oscillogram 15, the above oscillogram illustrates the relatively large transient effect caused by closing the circuit at the point of maximum e.

e = 60-cycle applied emf. E (eff.) = 35 volts.
i = current variation. I (eff.) = 0 27 amp.
R = 2.0 ohms. L = 0 012 henry. C = 19.8μf.
$\lambda = 90°$. $Q_0 = 0$.

Power Transients.—The general expression for instantaneous power in a *RLC* circuit is very unwieldly in form. Multiplying the current, equation (61), by $[E_m \sin(\omega t + \lambda)]$ yields the general expression for the power delivered to the circuit:

$$p = \frac{E_m^2}{2Z}\left[\cos\theta - \cos 2\left(\omega t + \lambda - \frac{\theta}{2}\right)\right] +$$

$$\epsilon^{-at}\left[\frac{E_m E_d'}{bL}\frac{(\epsilon^{bt} - \epsilon^{-bt})}{2} - \frac{E_m^2}{Z}\sin(\lambda - \theta)\frac{(\epsilon^{bt} + \epsilon^{-bt})}{2}\right]\sin(\omega t + \lambda) \quad (96)$$

Depending upon the value of b, the transient term assumes forms which correspond to those which have been given for i_t. Only the oscillatory case will be considered here. When b is imaginary:

$$p = \frac{E_m^2}{2Z}\left[\cos\theta - \cos 2\left(\omega t + \lambda - \frac{\theta}{2}\right)\right] + E_m I_t'[\sin(\beta t - \sigma)\sin(\omega t + \lambda)]\epsilon^{-at} \quad (97)$$

The number of factors involved is large, and close interrelationships exist between most of them. Nevertheless, equation (97) expresses the instantaneous power as a function of time, and the various terms of that equation may, to some extent, be interpreted. The first bracket term on the right-hand side of equation (97) is the well-known expression for the steady-state power. It is a double-frequency cosine variation which, if graphed, alternates above and below the $\left[\frac{E_m^2}{2Z}\cos\theta\right]$ ordinate. $\frac{E_m^2}{2Z}\cos\theta$ is the value of the average steady-state power. A brief review of the power variations under steady-state conditions will be given. This may help the reader to appreciate better the power oscillograms which are shown in this article.

$$i_s = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) \quad (98)$$

The counter-voltages are:

$$Ri_s = \frac{E_m R}{Z}\sin(\omega t + \lambda - \theta) \quad (99)$$

$$L\frac{di_s}{dt} = \frac{E_m L\omega}{Z}\cos(\omega t + \lambda - \theta) \quad (100)$$

$$\frac{q_s}{C} = -\frac{E_m}{\omega C Z}\cos(\omega t + \lambda - \theta) \quad (101)$$

The rates at which energy is transformed into heat, stored in the magnetic field, and stored in the electrostatic field are respectively:

$$p_R = \frac{E_m^2 R}{Z^2}\sin^2(\omega t + \lambda - \theta) \quad (102)$$

$$p_L = \frac{E_m^2 \omega L}{2Z^2}\sin 2(\omega t + \lambda - \theta) \quad (103)$$

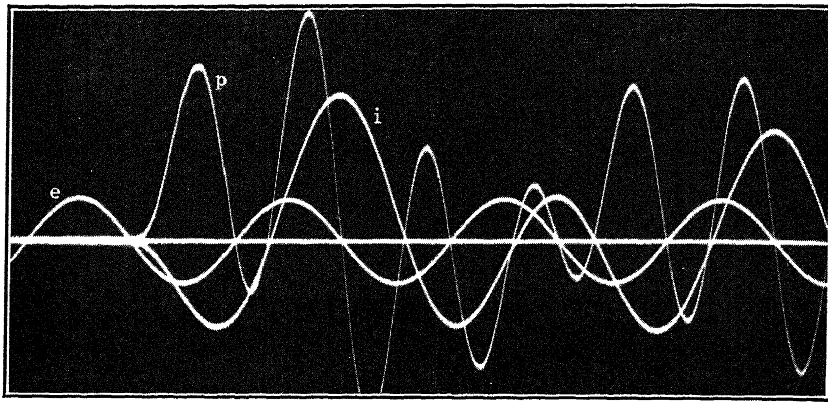
$$p_C = -\frac{E_m^2}{2\omega C Z^2}\sin 2(\omega t + \lambda - \theta) \quad (104)$$

In the above form the nature of the variations is at once apparent. In each case, the frequency is double that of the applied voltage. p_L and p_C are symmetrical variations about the $p = 0$ ordinate. They are exactly opposed to each other at all times. This follows directly from the fact that, during the parts of a cycle in which energy is being stored in the magnetic field, energy is being released from the electrostatic field, and vice versa. If $\omega L > \frac{1}{\omega C}$, the circuit is predominately inductive and the electrostatic field may be considered to be supplied with energy from that released by the magnetic field. Likewise, if $\frac{1}{\omega C} > \omega L$, the magnetic field may be thought of as receiving its energy from that released by the electrostatic field. Excepting the singular case of $\omega L = \frac{1}{\omega C}$, the *RLC* circuit, under steady-state conditions, simulates the behavior of either an *RL* or an *RC* circuit.

The second bracket term in equation (97) represents the transient component of the complete expression for instantaneous power. Except for the singular conditions which render I_t' equal to zero, this component is always present and persists until steady-state conditions are reached. It will be observed that the transient component is the product of two sinusoidal terms which, in general, have different frequencies. The circuit possesses two reservoirs for electrical energy, and many transient interchanges may be required before steady-state relations are obtained.

The nature of the power variation near $t = 0$ is largely dependent upon the relative values of β and ω. Equation (97) shows how the transient component of power depends upon these two factors. The power supplied to the circuit under the conditions of $\beta > \omega$ is illustrated by Oscillogram 17. A large number of energy interchanges are plainly shown. Steady-state conditions are not in evidence on this oscillogram, but the nature of the steady-state power variation is similar to that of a highly inductive circuit. Such has previously been shown.

The singular case in which $\beta = \omega$ is considered next. The manner in which the current attains its steady-state value has already been discussed. Oscillogram 18 illustrates the gradual and systematic manner in which steady-state power values are obtained. For $\beta = \omega$, the transient component of power acquires a frequency of variation which is equal to that of the steady component, namely, twice the frequency of the applied emf. An examination of equation (97) discloses the relationships that are involved. It is interesting to note that within the limits of oscillographic accuracy the circuit never possesses energy in

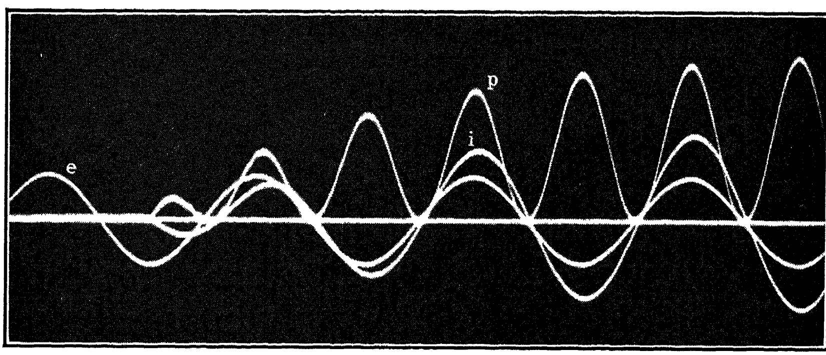


OSCILLOGRAM 17.

Voltage, current, and power in the *RLC* circuit. $\beta < \omega$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. I (eff.) = 4.2 amp.
p = power variation. P (avg.) = 70 watts.

$R = 4$ ohms. $L = 0.094$ henry. $C = 320\mu$f.
$\lambda = 185°$, approximately. $Q_0 = 0$.



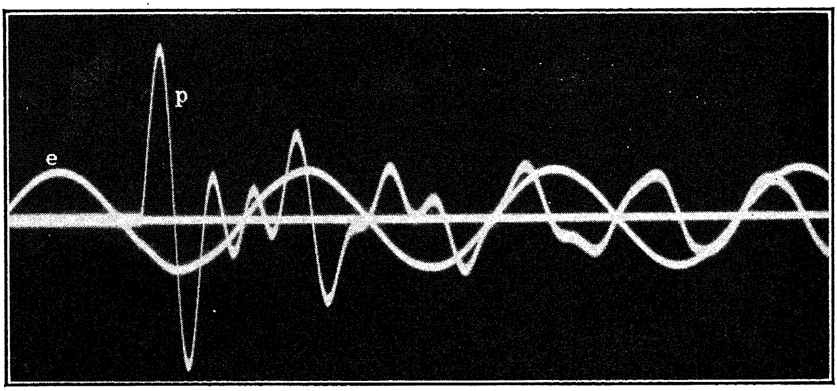
OSCILLOGRAM 18.

Voltage, current, and power in the *RLC* circuit. $\beta = \omega$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 44 volts.
i = current variation. I (eff.) = 4.7 amp.
p = power variation. P (avg.) = 210 watts.

$R = 9.3$ ohms. $L = 0.094$ henry. $C = 75\mu$f.
$\lambda = -90°$, approximately. $Q_0 = 0$.

excess of its own requirements. No energy is returned to the source. The energy interchanges between the magnetic and electrostatic fields are so balanced that they supply each other. And even under steady-state conditions where the energy interchanges between these two reservoirs are relatively large, no external evidence of these interchanges can be observed. Equations (103) and (104) show that p_L and p_C are equal and opposite when $\omega = \sqrt{\frac{1}{LC}}$. For the case illustrated, $\beta = \sqrt{\frac{1}{LC}}$ within the limits of experimental error.



OSCILLOGRAM 19.

Voltage and power in the *RLC* circuit. $\beta > \omega$.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
p = power variation. P (avg.) = 2 watts.
R = 2.0 ohms. L = 0.012 henry. C = 29.9μf.
λ = 210°, approximately. $Q_0 = 0$.

Oscillogram 19 illustrates the case of $\beta > \omega$. The current variation is purposely omitted on this particular oscillogram. The high-frequency variation in the power during the early period is, of course, due to the fact that β is relatively large as compared with ω. As steady-state conditions are reached, the power curve becomes a double-frequency variation with respect to the frequency of the applied emf. In the particular case illustrated the average steady-state power is very small as indicated by the relatively large negative power loops.

EXERCISES

1. Show why the maximum possible instantaneous value of the transient current in an RL circuit is less than twice the maximum steady-state current. It is assumed that R and L are constant.

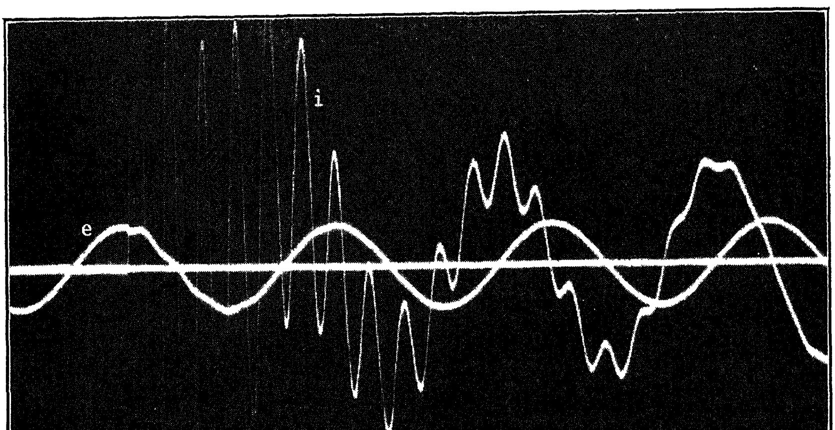
2. Show why the effective value of the transient current in the RL circuit cannot be as great as $\sqrt{3}$ times the effective value of the steady-state current.

3. (a) Solve equation (39) for i, and compare the result with the general expression for current in the RC circuit that is given in (38).

(b) Derive the expression for q in the RC circuit, and make a graph of the variation of q with respect to time.

4. Write the equation for the current graph shown in Oscillogram 2.

5. Write the equation for the current graph shown in Oscillogram 7.



OSCILLOGRAM 20.

Oscillogram to be used in connection with Exercise 6.

6. The larger graph of Oscillogram 20 is a photographic record of the transient current in a given RLC series circuit. The frequency of the applied voltage, indicated by the smaller graph, is 60 cycles per second.

(a) What is the value of λ?

(b) Is the circuit predominately inductive or capacitive?

(c) Which of the three cases of transient current is represented by the oscillogram?

(d) What is the approximate frequency of the transient component of the current?

(e) Write the generalized expression for the current graph.

(f) If $E = 115$ volts effective, $R = 2$ ohms, $L = 0.012$ henry, $C = 15$ microfarads, and $Q_0 = 0$, write the equation for the current graph employing numerical coefficients.

(g) Estimate the relative maximum values of the transient and steady-state components of current from the above equation.

7. A 60-cycle alternating voltage having a maximum magnitude of 3250 volts is suddenly applied to a series circuit of the RL type. $R = 21.75$ ohms and $L = 0.1$ henry. Write the equation for i, employing numerical coefficients. What will be the maximum value of current, and at what time will it occur? Let it be assumed that the circuit is closed 0.00139 second before the beginning of a cycle on the voltage wave.

8. A 60-cycle alternating voltage having a maximum magnitude of 3250 volts is suddenly applied to a series circuit of the RC type. $R = 32.5$ ohms and $C = 200$ microfarads. Assuming that the circuit is closed at the point of maximum positive voltage, write the equation for i, employing numerical coefficients. What will be the maximum value of the current and at what time will it occur? $Q_0 = 0$.

9. Write the equations for the current variation and the power variation shown in Oscillogram 17. What is the maximum value of current that is indicated on the oscillogram? What is the maximum value of power that is indicated? What is the approximate value of the average power during the first 1/60th of a second after $t = 0$? What is the effective value of the current during the first 1/60th of a second after $t = 0$? Compare the average power and the effective current during the first cycle with the corresponding values of average power and effective current during the steady-state period.

10. A 220,000-volt, three-phase, 25-cycle transmission line has the following parameters (assumed constant). The resistance of each line wire is 28 ohms, the inductive reactance is 80 ohms, and the capacitive reactance is 2000 ohms to neutral. Assume that the line is open at the receiving end and that all the capacitance is concentrated at the middle of the line. A voltage to neutral which is represented by $e = 179{,}600 \cos \omega t$ is impressed upon the line at $t = 0$. Plot the line current graph for one cycle of the voltage variation. Show the steady-state term, the transient term, and the resultant current on the same sheet.

Repeat for $e = 179{,}600 \cos (\omega t - 90°)$ impressed.

11. A balanced three-phase, wye-connected load consisting of an impedance of $22.3 + j322.4$ ohms per leg is connected to a balanced, 60-cycle, three-phase source of voltage at $t = 0$. The line-to-line voltage is equal to 220 volts effective. The switching operation is performed at the time when the instantaneous voltage of phase $n1$ is equal to $+118$ volts and $\frac{de}{dt}$ is positive. Assume that the phase rotation is $n1-n2-n3$. Plot each of the three load currents for at least two complete cycles of the voltage variation. It is suggested that the plots be placed one above the other so that the time scales coincide. Indicate the steady-state terms and the transient terms in dotted lines.

CHAPTER VIII

PARTICULAR BOUNDARY CONDITIONS

The alternating-current circuits which have been considered up to this point have been assumed to be at rest prior to $t = 0$. The case of initial condenser charge has been considered, but the current in the circuit prior to the switching operation has in every instance been assumed to be of zero value. If a current is present just prior to $t = 0$ the circuit will, in general, possess a kinetic energy equal to $LI_0^2/2$, and this energy must be returned to the circuit before new steady-state conditions can be established. The above phenomenon may be illustrated by considering particular examples of the transition in current and power from one set of values to another.

CHANGE IN EXCITING VOLTAGE

Example 1.—Consider the simple case of an RL circuit carrying a steady direct current prior to $t = 0$. Let this current be known as I_0, and assume that, at $t = 0$, the battery voltage is disconnected and that an alternating potential difference is applied to the circuit. Fig. 1 is a schematic diagram of the circuit arrangement. It will be assumed that switch s can be changed from position a to position b instantaneously.

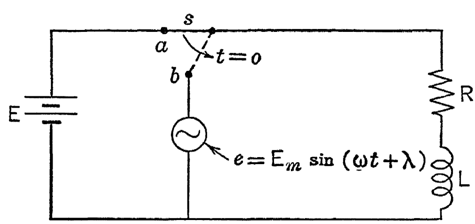


FIG. 1.—The exciting voltage across the RL branch is changed from E to $E_m \sin(\omega t + \lambda)$ at $t = 0$.

The expression for the current at, and after, $t = 0$ is:

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + \left[I_0 - \frac{E_m}{Z}\sin(\lambda - \theta)\right]\epsilon^{-\frac{Rt}{L}} \qquad (1)$$

$I_0 = \dfrac{E_{dc}}{R}$, it being assumed that the current, with the switch at position a, is at its steady-state value.

The derivation and physical interpretation of equation (1) follow directly from the analysis given in Chapter VII. A is evaluated as follows:

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + A \epsilon^{-\frac{Rt}{L}} \tag{2}$$

At $t = 0$:

$$i = I_0$$

Therefore:

$$A = \left[I_0 - \frac{E_m}{Z} \sin (\lambda - \theta) \right] \tag{3}$$

A is the value of the transient component of the current at $t = 0$. The steady direct current, I_0, is a part of that factor. As the transient

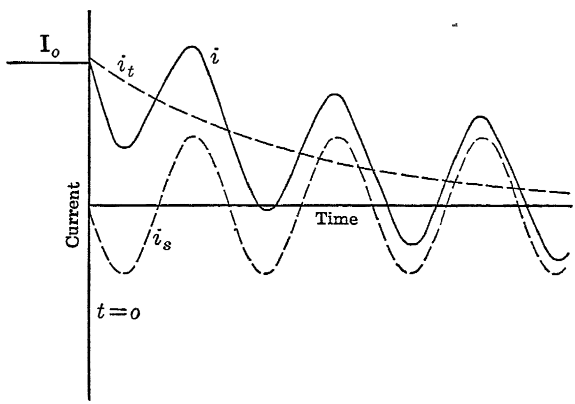
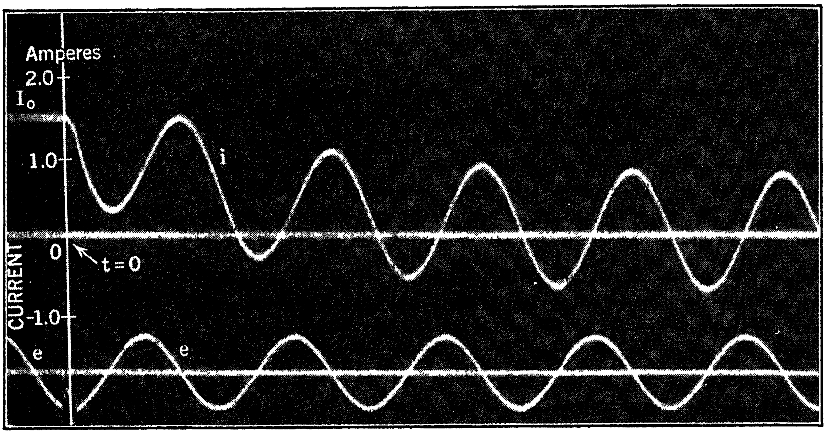


FIG. 2.—Plotted solution of equation (1) for a particular case.
I_0 = original continuous current prior to $t = 0$.
i_t = transient component of current after $t = 0$.
i_s = steady-state component of current after $t = 0$.
$i = i_t + i_s$, resultant current after $t = 0$.
$E_{d\text{-}c} = 6.2$ volts. $e_{a\text{-}c} = 25\ 5 \sin (377t - 90°)$.
$R = 4.1$ ohms. $L = 0.094$ henry.

component subsides the actual current in the circuit changes from I_0 to $(E_m/Z) \sin (\omega t + \lambda - \theta)$, its steady-state alternating-current value. During the transition period I_0 subsides, exponentially, and in so doing transfers the kinetic energy, $LI_0/2$, from the magnetic field to the electric circuit proper.

Oscillographic Demonstration.—Fig. 2 shows the plotted solution of the various component currents, namely, I_0, i_t, and i_s. The manner in which i_t and i_s combine to form the actual current in the circuit after $t = 0$ is illustrated. The boundary conditions and the circuit parame-

ters which have been selected are similar to those under which Oscillogram 1 was taken. Actually, to switch from a low-resistance battery source to an alternating source of potential difference as indicated in Fig. 1 requires very precise switching apparatus. Oscillogram 1 shows quite clearly that the alternator was short-circuited for a period of approximately 0.001 second just prior to the release of the battery potential. The alternating source of potential quickly recovered itself, however, and no marked reflections appear in the current variation.



OSCILLOGRAM 1.

Illustrating the transition from direct current to steady-state alternating current. See Fig. 1

$e = 25.5 \sin (377t - 90°)$, applied alternating potential.

i = current variation. $I_0 = 1.5$ amp.

$R = 4.1$ ohms. $L = 0.094$ henry.

Example 2.—In this example the change from one alternating voltage, e_a, to another alternating voltage, e_b, is considered. Prior to $t = 0$ the RL circuit shown in Fig. 3 is assumed to be carrying a steady alternating current which is supplied by alternator a. Under these conditions,

$$i_a = \frac{E_{ma}}{Z_a} \sin (\omega_a t_a) \tag{4}$$

t_a is reckoned from the beginning of a cycle, namely from the point at which $i_a = 0$ and $\dfrac{di_a}{dt}$ positive. ω_a is the angular velocity of e_a. i_a may, of course, have any value between $-E_{ma}/Z_a$ and $+E_{ma}/Z_a$. The energy stored in the magnetic field is therefore dependent upon the value of t_a.

At $t = 0$, it is assumed that e_a is disconnected and that e_b is connected to the RL circuit. The point, $t = 0$, may be at any value of t_a.

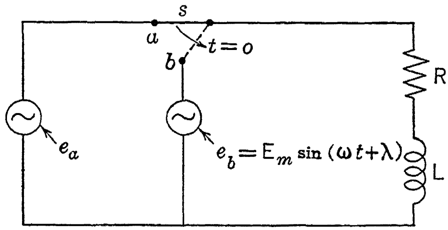


FIG. 3.—Change from one alternating potential to another at $t = 0$.

In order to simplify the expression for current, it is desirable that this new point of reference be introduced.

Let:

$$i_a \text{ (at } t = 0) = i_0$$

$$e_b = E_{mb} \sin (\omega t + \lambda) \tag{5}$$

Then:

$$i_b = \frac{E_{mb}}{Z} \sin (\omega t + \lambda - \theta) + \left[i_0 - \frac{E_{mb}}{Z} \sin (\lambda - \theta) \right] \epsilon^{-\frac{Rt}{L}} \tag{6}$$

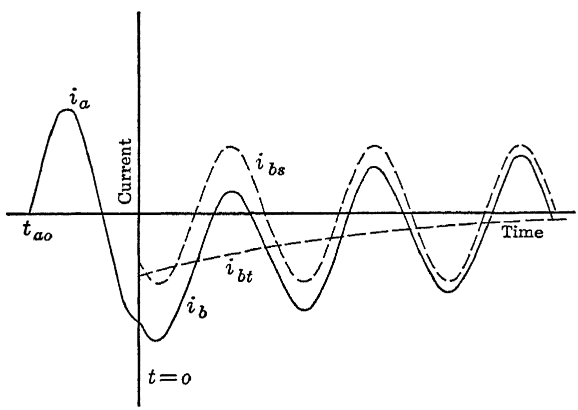


FIG. 4.—Plotted solution of equation (6) for a particular case.

$i_a = \dfrac{78}{35.5} \sin 377t_a$ (t_a counted from t_{ao}).

i_{bt}, transient component of i_b.

i_{bs}, steady-state component of i_b.

$i_b = i_{bs} + i_{bt}$, the resultant current after $t = 0$.

$e_a = 78 \sin (377t_a + 83°)$ not shown.

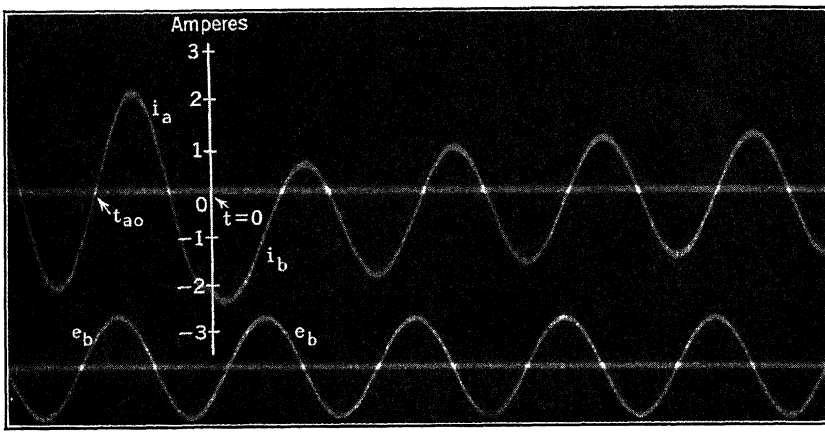
$e_b = 50 \sin (377t - 55°)$ not shown.

$R = 4$ ohms. $L = 0.094$ henry. $\lambda = -55°$.

i_b is the current which flows in the circuit after $t = 0$. λ defines the point on the e_b wave at which the switching operation is performed.

Oscillographic Demonstration.—Oscillogram 2 illustrates the transition of current from one set of values to another. It will be observed that the switching operation is performed at the point of negative maximum i_a. $t = 0$ is also indicated on the e_b wave; however, no serious disturbance to e_b is noticeable as a result of the switching operation.

The analytically determined components of i_b are shown graphically in Fig. 4. They serve to illustrate the theoretical considerations in a



OSCILLOGRAM 2.

Illustrating the transition in current from one set of alternating current values to another in the RL circuit. See Fig. 3.

$$i_a = \frac{78}{35.5} \sin 377t_a \text{ (current prior to } t = 0\text{)}.$$

i_b = resultant current after the application of e_b.

$e_b = 50 \sin (377t - 55°)$. (Time counted from $t = 0$.)

$R = 4$ ohms. $L = 0.094$ henry.

more detailed manner than is possible oscillographically. Comparison of Oscillogram 2 and the resultant current graph of Fig. 4 will show the correspondence that exists between experimental and theoretical values.

CHANGE IN CIRCUIT PARAMETERS

A sudden change in the circuit parameters will, of course, result in a change of current from one set of values to another. Consider the general case of an RLC series circuit that is energized by an alternating potential. At $t = 0$ it will be assumed that a switching operation is

performed which will suddenly alter the parameters of the circuit. Fig. 5 is a schematic diagram of an arrangement in which part of the circuit parameters are short-circuited at $t = 0$. The circuit is originally carrying a current which will be known as i_a.

$$i_a = \frac{E_m}{Z_a} \sin \omega t_a \tag{7}$$

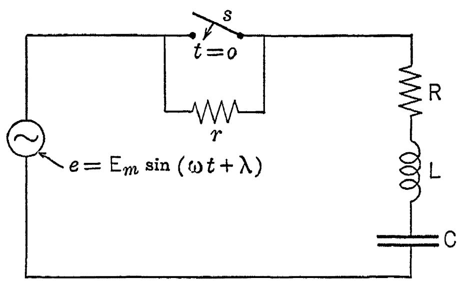


FIG 5.—Change in circuit parameters at $t = 0$.

Z_a is the impedance of the circuit prior to $t = 0$.

t_a is reckoned from the beginning of a cycle of the steady-state variation of i_a just prior to $t = 0$.

$$i_a = \frac{dq_a}{dt}.$$

From which:

$$q_a = -\frac{E_m}{\omega Z_a} \cos \omega t_a, \text{ under steady-state conditions.} \tag{8}$$

Mathematical Development.—It will be remembered that the general expression for dynamic equilibrium in the RLC circuit is:

$$Ri + L\frac{di}{dt} + \frac{q}{C} = E_m \sin (\omega t + \lambda) \tag{9}$$

The general expression for the current has been shown to be:

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + c_1 \epsilon^{\alpha_1 t} + c_2 \epsilon^{\alpha_2 t} \tag{10}$$

where $\alpha_1, \alpha_2 = -a \pm b$.

$$i = i_0 = \frac{E_m}{Z_a} \sin \delta, \text{ at } t = 0 \tag{11}$$

δ is the value which ωt_a has at the time of performing the switching operation.

$$q = Q_0 = -\frac{E_m}{\omega Z_a} \cos \delta, \text{ at } t = 0 \tag{12}$$

The constants c_1 and c_2 of equation (10) may be evaluated in terms of the boundary conditions stated in (11) and (12). From equation (9):

$$\left(\frac{di}{dt}\right)_{t=0} = \frac{1}{L}\left[E_m \sin\lambda - Ri_0 - \frac{Q_0}{C}\right] \tag{13}$$

From equation (10):

$$\left(\frac{di}{dt}\right)_{t=0} = \frac{E_m\omega}{Z}\cos(\lambda - \theta) + c_1\alpha_1 + c_2\alpha_2 \tag{14}$$

Therefore:

$$c_1\alpha_1 + c_2\alpha_2 = \frac{1}{L}\left[E_m \sin\lambda - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - Ri_0 - \frac{Q_0}{C}\right] \tag{15}$$

$$c_1 + c_2 = i_0 - \frac{E_m}{Z}\sin(\lambda - \theta) \tag{16}$$

Solving (15) and (16) simultaneously and substituting for α_1 and α_2 results in:

$$c_1 = \frac{1}{2bL}\left[E_m \sin\lambda - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - \frac{E_m R}{2Z}\sin(\lambda - \theta) - \frac{Ri_0}{2} - \frac{Q_0}{C}\right] + \left[\frac{i_0}{2} - \frac{E_m}{2Z}\sin(\lambda - \theta)\right] \tag{17}$$

$$c_2 = -\frac{1}{2bL}\left[E_m \sin\lambda - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - \frac{E_m R}{2Z}\sin(\lambda - \theta) - \frac{Ri_0}{2} - \frac{Q_0}{C}\right] + \left[\frac{i_0}{2} - \frac{E_m}{2Z}\sin(\lambda - \theta)\right] \tag{18}$$

The values of c_1 and c_2 differ from those derived under the condition of zero initial current only by the i_0 factors. The details involved in arriving at the following expression for current, in the oscillatory case are similar to those previously outlined.

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + I_t'' \sin(\beta t + \sigma)\epsilon^{-at} \tag{19}$$

$$I_t'' = \sqrt{\left[\frac{E_d''}{\beta L}\right]^2 + \left[i_0 - \frac{E_m}{Z}\sin(\lambda - \theta)\right]^2} \tag{20}$$

$$E_d'' = \left[E_m \sin\lambda - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - \frac{E_m R}{2Z}\sin(\lambda - \theta) - \frac{Ri_0}{2} - \frac{Q_0}{C}\right] \tag{21}$$

$$\sigma = \tan^{-1} \frac{\beta L\left[i_0 - \frac{E_m}{Z} \sin (\lambda - \theta)\right]}{E_d''} \tag{22}$$

The other symbols have the customary meanings. For convenience:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

R, L, and C are the circuit parameters after the switching operation has been performed.

$$\theta = \tan^{-1} \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

i_0 is the value of the current in the original circuit at $t = 0$. See equation (11).

$$a = \frac{R}{2L}$$

$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Equation (19) defines the current variation at the time of and for the period following a sudden alteration of the circuit parameters. Detailed examination of the expression will show that i starts at the value i_0 at $t = 0$. It will also reveal the manner in which the natural behavior of the circuit manifests itself during the period immediately following $t = 0$. After a lapse of time the circuit acquires its steady-state variation, namely,

$$i_s = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta)$$

Example 1.—The plotted solutions of i_s and i_t for a particular case are shown in Fig. 6. The i_t variation illustrates the circuit's natural behavior under a particular set of boundary conditions. It will be observed, from the legend given below the graphs, that the case has been selected more or less at random. The circuit's natural damped angular velocity is 595 radians per second as compared with 377 radians per second, the angular velocity of the driving voltage. The resultant current is therefore the sum of a 95-cycle transient component and a 60-cycle steady-state component. The sum of i_s and i_t is the actual

current after $t = 0$. The alteration in circuit parameters in this case is a change in the total resistance of the circuit at $t = 0$. Reference to Fig. 6 will show that the alteration in the resistance of the circuit is from $(r + R)$ to R.

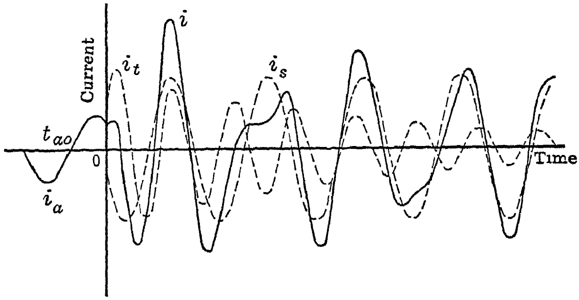


FIG. 6.—Plotted solution of equation (19) for the particular case of changing the resistance from $(r + R)$ to R.

$$i_a = \frac{163}{114} \sin 377 t_a \text{ (counting } t_a \text{ from } t_{ao})$$

$$i_s = \frac{163}{53.1} \sin (377t + 201°) \quad \text{(counting } t \text{ from } t = 0).$$

$$i_t = 3.38 \sin (595t + 36°)\, \epsilon^{-21.3t}.$$

$$i = i_s + i_t, \text{ the resultant current in the circuit.}$$

$r = 97$ ohms. $R = 4$ ohms. $L = 0.094$ henry. $C = 30\mu$f. $\lambda = 115°$.

Oscillographic Demonstration.—Oscillogram 3 was taken under approximately the same boundary conditions and with the same circuit parameters as those employed in making the plotted solutions shown in

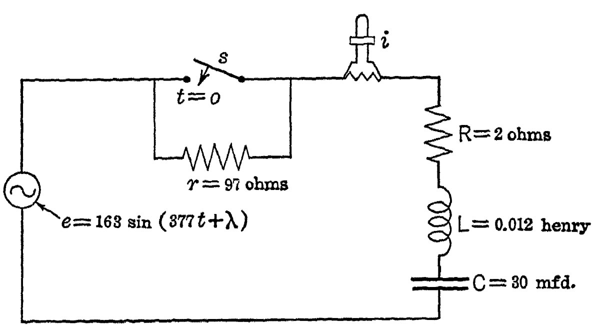
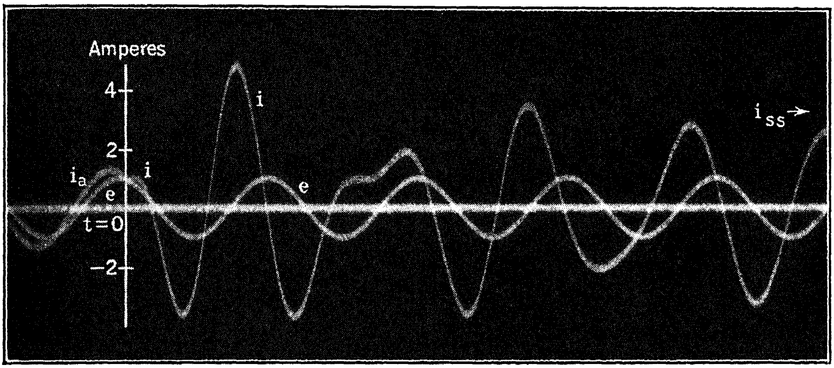


FIG. 7.—Oscillogram 4 illustrates the effect of suddenly eliminating r at $t = 0$.

Fig. 6. It illustrates the actual variation in the current before and after $t = 0$. Short-circuiting 97 ohms has caused an appreciable change in the magnitude of i. Another noticeable feature is the change in the phase position of the current with respect to the applied voltage.

Example 2.—A case similar in nature to the one just described is illustrated by Oscillogram 4. The alteration which is made in the circuit parameters is evident from the arrangement shown in Fig. 7.

The chief distinction of the present example is the relatively high value of β. The parameters are essentially the same as in the preceding example except for the self-inductance. Lowering L raises the natural frequency of the circuit, and the oscillogram shows quite plainly that the natural frequency is approximately five times that of the steady-state frequency. Calculations show that β is 1670 radians per second



OSCILLOGRAM 3.

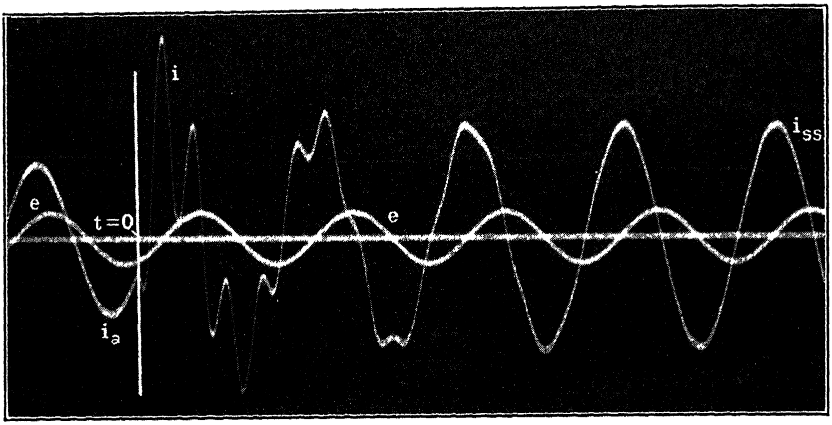
Transition in current in the *RLC* circuit caused by the elimination of a portion of the circuit's resistance. See Fig. 5.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. i_a (eff.) = 1 amp. (before $t = 0$).
i_{ss} (eff.) = 2.12 amp. (after $t = 0$).
r = 97 ohms. R = 4 ohms. L = 0.094 henry. $C = 30\mu$f.
$\lambda = 115°$, approximately.
r is eliminated by short-circuiting at $t = 0$.

as compared with ω equal to 377. The complete expression for the current might be written according to the method previously outlined. The oscillogram is shown for the purpose of further illustrating the fact that, during periods of transition, there is a marked exhibition of the circuit's natural behavior. It will be observed that the transient term subsides at a considerably faster rate than in Oscillogram 3 on account of the increase in the ratio of R to L.

Example 3.—The transition of current possesses a rather marked peculiarity when the natural period of the circuit coincides with that of the driving voltage. Oscillogram 12 in the preceding chapter shows the gradual manner in which the current changes from zero to its steady-

state value. The transition here considered is from one set of alternating-current values to another. Originally the circuit is of a highly inductive nature. At $t = 0$ a certain amount of self-inductance is



OSCILLOGRAM 4.

Transition in current in the RLC circuit caused by the elimination of a portion of the circuit's resistance. See Fig. 7.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. i_a (eff.) = 0 87 amp. (before $t = 0$).
i_{ss} (eff.) = 1.3 amp. (after $t = 0$).
r = 97 ohms. R = 2 ohms. L = 0 012 henry. C = 30μf.
r is eliminated by short-circuiting at $t = 0$.

short-circuited. The remaining circuit parameters are such that β is approximately equal to ω. The arrangement may be better understood by referring to Fig. 8.

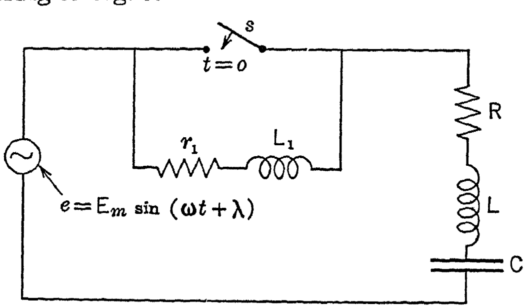
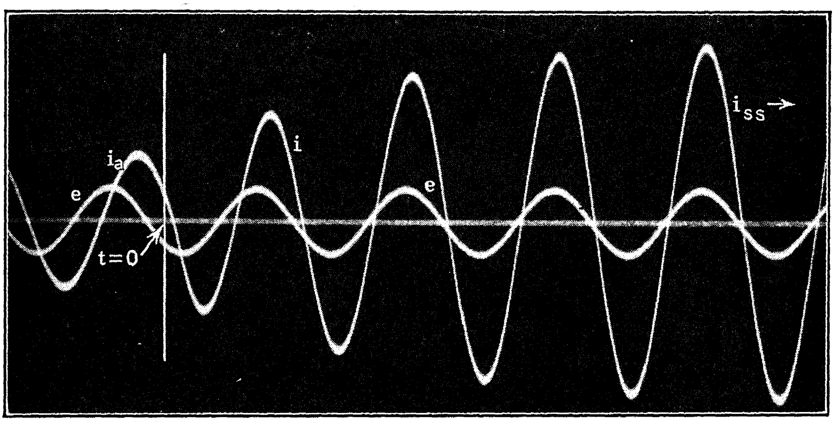


FIG. 8.—Oscillogram 5 illustrates the effect of short-circuiting r_1 and L_1.

Oscillogram 5 represents the plotted solution of the actual current variation during the period of transition. The original current is

shown for only about one cycle prior to $t = 0$. The fact that the frequency of the transient component is practically equal to that of the steady-state component accounts for the smoothness of the transition. Equation (19) is useful in helping to visualize the component variations. Under the conditions of $\beta = \omega$ the actual current in the circuit will possess no marked irregularities. An exponentially damped sine term and a sine term of like frequency combine to give the smooth variation shown by the oscillogram. There is a gradual shift in the



OSCILLOGRAM 5.

The transition in current in the RLC circuit when $\beta = \omega$. See Fig. 8.

e = 60-cycle sinusoidal applied emf. E (eff.) = 29 volts.
i = current variation. i_a (eff.) = 1.05 amp. (before $t = 0$).
i_{ss} (eff.) = 2.85 amp. (after $t = 0$).

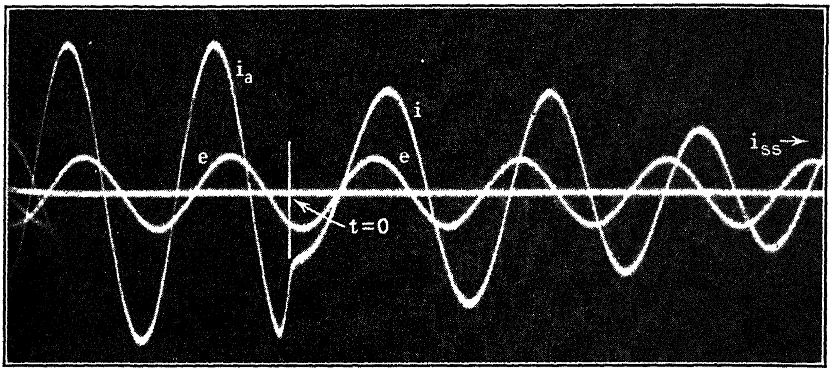
$r_1 = 2.65$ ohms. $L_1 = 0.066$ henry. $R = 10$ ohms. $L = 0.094$ henry. $C = 75\mu$f.
r_1 and L_1 are short-circuited at $t = 0$.

time phase of the current with respect to the applied voltage throughout the period of transition.

Example 4.—The phenomena which accompany the insertion of circuit parameters are of the same general nature as those discussed in Examples 1, 2, and 3. Consider the circuit arrangement shown in Fig. 9.

The case is chosen to illustrate the rather marked change in the phase position of the current with respect to the applied voltage. The original impedance of the circuit is predominantly capacitive so that i_a leads the voltage by approximately 45 degrees. At $t = 0$, a relatively large inductance is added, with the result that the impedance of the circuit becomes predominantly inductive, so that under

steady-state conditions the current will lag the voltage by approximately 65 degrees. The shift in the phase position of the current from its leading to lagging position is clearly shown in Oscillogram 6.



OSCILLOGRAM 6.

Illustrating the change from a leading current to a lagging current due to an alteration in the circuit parameters. See Fig. 9.

e = 60-cycle sinusoidal applied emf. E (eff.) = 31 volts.
i = current variation. i_a (eff.) = 2.3 amp. (before $t = 0$).
i_{ss} (eff.) = 1.1 amp. (after $t = 0$).

r_1 = 8.5 ohms. L_1 = 0.066 henry. R = 11 7 ohms. L = 0.16 henry. $C = 75\mu$f
R, L, and C are the circuit parameters after r_1 and L_1 are added at $t = 0$.

OSCILLOGRAMS OF TRANSITIONS IN POWER.—The general trends in the power variation are evident from the expression which has been

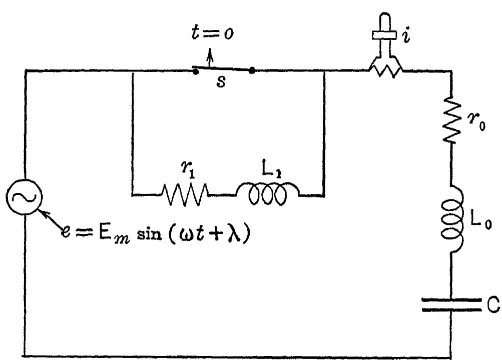
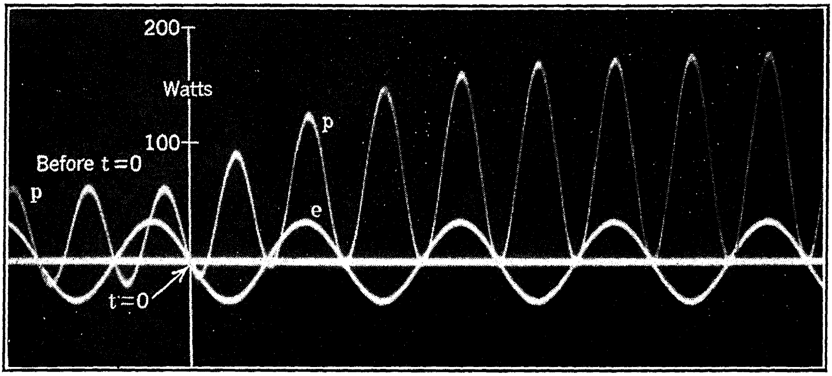


FIG. 9.—Oscillogram 6 illustrates the effect of suddenly introducing r_1 and L_1 into the circuit.

derived for the instantaneous current during the period of transition. Only two particular cases will be considered.

Example 1.—The transition in power from one set of steady-state values to another is shown in Oscillogram 7. Consideration of the



OSCILLOGRAM 7

Illustrating the gradual transition in power in the *RLC* circuit when $\beta = \omega$. See Fig. 10.

e = 60-cycle sinusoidal applied emf. E (eff.) = 35 volts.
p = power variation. P (avg.) = 20 watts (before $t = 0$).
P_{ss} (avg.) = 90 watts (after $t = 0$).

$L_1 = 0.066$ henry. $r_1 = 2.65$ ohms. $R = 13$ ohms. $L = 0.094$ henry. $C = 75\mu$f.
L_1 and r_1 are eliminated from the circuit at $t = 0$.

relative magnitudes of the parameters will reveal the reason for the smoothness of that transition. After the switching operation has

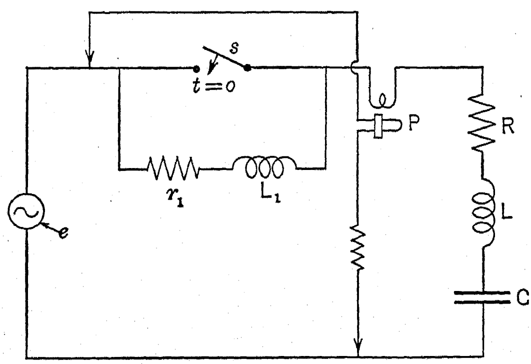
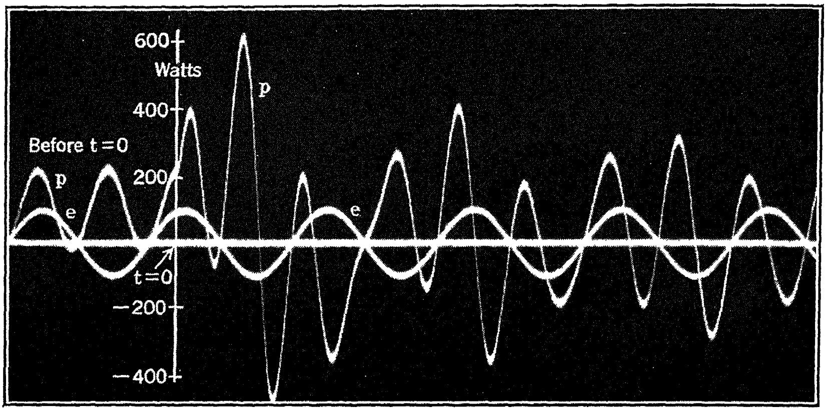


FIG. 10.—Circuit arrangement employed in obtaining the power transient shown in Oscillogram 7.

been performed the natural frequency of the circuit, $\frac{\beta}{2\pi}$, is equal to the frequency of the applied emf.

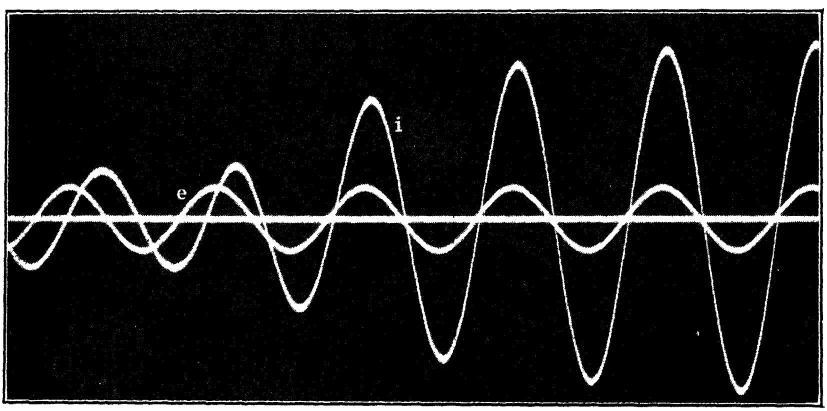


OSCILLOGRAM 8

Transition in power in the RLC circuit caused by an alteration in the circuit parameters.

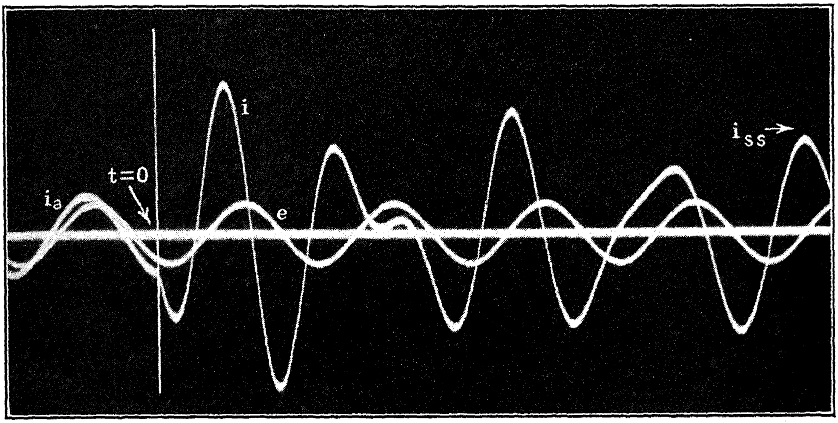
e = 60-cycle sinusoidal applied emf. E (eff.) = 114 volts.
p = power variation. P (avg.) = 125 watts (before $t = 0$).
P_{ss} (avg.) = 25 watts (after $t = 0$).

r_1 = 70 ohms. R = 3 ohms. L = 0.066 henry. C = 37.5μf.
r_1 is eliminated at $t = 0$. $\lambda = 90°$, approximately.



OSCILLOGRAM 9.

To be used in connection with Exercise 1.

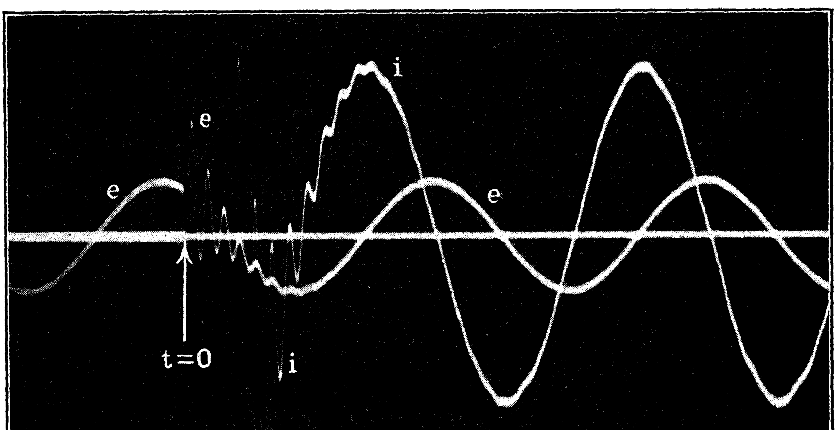


OSCILLOGRAM 10.

To be used in connection with Exercise 3.

e = 60-cycle sinusoidal applied emf. E (eff.) = 115 volts.
i = current variation. i_a (eff.) = 1 amp. (before $t = 0$).
i_{ss} (eff.) = 2.12 amp. (after $t = 0$).

$r = 97$ ohms. $R = 4$ ohms. $L = 0.094$ henry. $C = 30\mu$f.
r is short-circuited at $t = 0$. $\lambda = -125°$, approximately.



OSCILLOGRAM 11.

To be used in connection with Exercise 5.

e = the line-to-line voltage generated by a 60-cycle, three-phase, 5-kilovolt-ampere sine wave alternator. Alternator on open circuit prior to $t = 0$. The effective steady-state value of the voltage is 110 volts.

i = the current delivered by the above alternator following a particular switching operation. The effective value of the steady-state current is approximately 1.1 amp.

Example 2.—As contrasted to the transition shown in the above example, Oscillogram 8 illustrates the irregular manner in which power transitions may occur. The point on the voltage wave at which the switching operation takes place is clearly shown by the point of discontinuity in the power variation. The relatively high natural frequency of the circuit is in evidence. It may be of interest to note that while the average power in this case changes from 125 watts originally to a new steady-state average of only 25 watts the instantaneous power reaches values of over 600 watts during the period of transition.

EXERCISES

1. What change in circuit parameters will produce the current transition shown in Oscillogram 9?

2. Consider a series RLC circuit in which $\frac{R^2}{4L^2}$ is negligibly small as compared with $\frac{1}{LC}$. Show that steady-state resonance obtains when the circuit parameters are so adjusted that $\beta = \omega$.

3. Write the equation of the current graph shown on Oscillogram 10, employing numerical coefficients Make a rough sketch of the steady-state and transient components, and compare $(i_s + i_t)$ with the oscillographic record.

4. Show that the term I_t'' as used in equation (19) is equal to

$$\sqrt{\left[\frac{E_d''}{\beta L}\right]^2 + \left[i_0 - \frac{E_m}{Z}\sin(\lambda - \theta)\right]^2}$$

Is the right-hand member of equation (20) dimensionally equivalent to current?

5. Refer to Oscillogram 11. To what can the violent circuit disturbance near $t = 0$ be attributed? How can the high-frequency variation in the voltage be explained?

CHAPTER IX

A-C TRANSIENTS IN DIVIDED CIRCUITS

The general mathematical predetermination of transient response in divided circuits with alternating voltage applied is not within the scope of this text. Elementary methods are not applicable except in simple cases.

PARALLEL BRANCHES

$R_1 L_1$ in Parallel with $R_2 C_2$.—With a circuit arrangement as shown in Fig. 1 the transient response of each individual branch may be

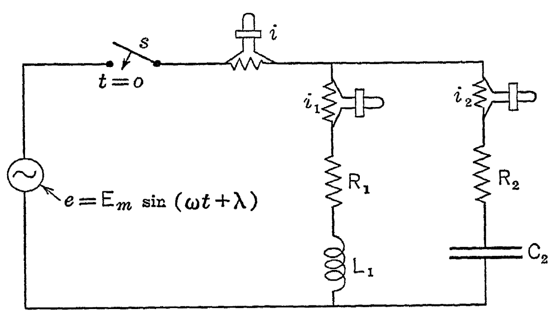


FIG. 1.—Parallel branches. Circuit arrangement employed in obtaining Oscillograms 1 and 2.

determined separately and the net effect obtained by applying Kirchhoff's current law. It has been shown in a previous chapter that:

$$i_1 = \frac{E_m}{Z_1}\left[\sin(\omega t + \lambda - \theta_1) - \epsilon^{-\frac{R_1 t}{L_1}} \sin(\lambda - \theta_1)\right] \qquad (1)$$

and

$$i_2 = \frac{E_m}{Z_2}\sin(\omega t + \lambda - \theta_2) + \frac{E_m}{R_2}\sin\theta_2 \cos(\lambda - \theta_2)\,\epsilon^{-\frac{t}{R_2 C_2}} \qquad (2)$$

where

$$Z_1 = \sqrt{R_1^2 + \omega^2 L_1^2}$$

$$\theta_1 = \tan^{-1}\frac{\omega L_1}{R_1}$$

$$Z_2 = \sqrt{R_2^2 + \frac{1}{\omega^2 C_2^2}}$$

$$\theta_2 = \tan^{-1} \frac{-1}{R_2 \omega C_2}$$

The current supplied to the two branches in parallel is the sum of i_1 and i_2.

$$i = i_1 + i_2 \tag{3}$$

It is evident that the transient effect in an individual branch of an electrical circuit of this kind is to add to the steady-state a-c term a second term which is simply an exponentially damped d-c component, the magnitude and duration of which depend upon the circuit parameters and the time at which the switch is closed.

The transient effects in the two branches of Fig. 1 may, for a particular set of parameters, cancel one another. For this condition to be attained it is necessary that the sum of the two transient terms be zero continuously after the closing of the switch s. Expressed mathematically, this means that:

$$\frac{E_m}{R_2} \sin \theta_2 \cos (\lambda - \theta_2) \epsilon^{-\frac{t}{R_2 C_2}} - \frac{E_m}{Z_1} \sin (\lambda - \theta_1) \epsilon^{-\frac{R_1 t}{L_1}} = 0 \tag{4}$$

The above equation is satisfied when the two terms are equal in magnitude but opposite in sign at $t = 0$ and are damped out at exactly the same rate. The latter condition obtains when:

$$\frac{1}{R_2 C_2} = \frac{R_1}{L_1}$$

A singular condition under which the above relation is satisfied is as follows:

$$R_1 = R_2 = \sqrt{\frac{L_1}{C_2}} = R \tag{5}$$

Under these conditions it may readily be shown that:

$$\sin \theta_2 = - \cos \theta_1 \tag{6}$$

and that:

$$\sin \theta_1 = + \cos \theta_2 \tag{7}$$

Therefore:

$$\frac{E_m}{R_2} \sin \theta_2 \cos (\lambda - \theta_2) = \frac{E_m}{Z_1} \sin (\lambda - \theta_1) \tag{8}$$

The transformations involved are:

$$\frac{E_m}{R_2} \sin \theta_2 \cos (\lambda - \theta_2) = \frac{E_m}{R_1} \cos \theta_1 \sin (\lambda - \theta_1) \tag{9}$$

$$\frac{E_m}{R} \sin \theta_2 \cos (\lambda - \theta_2) = \frac{E_m}{R} [- \sin \theta_2] [- \cos (\lambda - \theta_2)] \tag{10}$$

Under the conditions stated in (5) the transient components in the two branches are at all times equal and opposite and the combination simulates a purely resistive circuit. The total current flowing to the two branches in parallel is:

$$i = \frac{E_m}{Z_1} \sin (\omega t + \lambda - \theta_1) + \frac{E_m}{Z_2} \sin (\omega t + \lambda - \theta_2) \tag{11}$$

$$= \frac{E_m}{R_1} \cos \theta_1 \sin (\omega t + \lambda - \theta_1) + \frac{E_m}{R_2} \cos \theta_2 \sin (\omega t + \lambda - \theta_2) \tag{12}$$

$$= \frac{E_m}{R_1} \cos \theta_1 \sin (\omega t + \lambda - \theta_1) + \frac{E_m}{R_2} \sin \theta_1 \cos (\omega t + \lambda - \theta_1) \tag{13}$$

Since:

$$R_1 = R_2 = R$$

$$i = \frac{E_m}{R} \sin (\omega t + \lambda) \tag{14}$$

A Particular Case:

Given the following constants:

$$E_{\text{eff}} = 24.5 \text{ volts, 60-cycle.}$$

$$R_1 = 3.5 \text{ ohms.}$$

$$L_1 = 0.056 \text{ henry.}$$

$$R_2 = 6.9 \text{ ohms.}$$

$$C_2 = 168 \text{ microfarads.}$$

Then:

$$\theta_1 = \tan^{-1} \frac{\omega L_1}{R_1} = \tan^{-1} 6.03 = 80.°6$$

$$\theta_2 = \tan^{-1} \frac{-1}{R_2 C_2 \omega} = \tan^{-1} (-2.28) = -66.°4$$

$$\frac{R_1}{L_1} = \frac{3.5}{0.056} = 62.5$$

$$\frac{1}{R_2 C_2} = \frac{10^6}{6.9 \times 168} = 862$$

$$Z_1 = \sqrt{3.5^2 + 21.1^2} = 21.4 \text{ ohms}$$

$$Z_2 = \sqrt{6.9^2 + 15.8^2} = 17.2 \text{ ohms}$$

Equations (1) and (2) become:

$$i_1 = \frac{24.5\sqrt{2}}{21.4}[\sin(377t + \lambda - 80.°6) - \epsilon^{-62.5t}\sin(\lambda - 80.°6)] \tag{15}$$

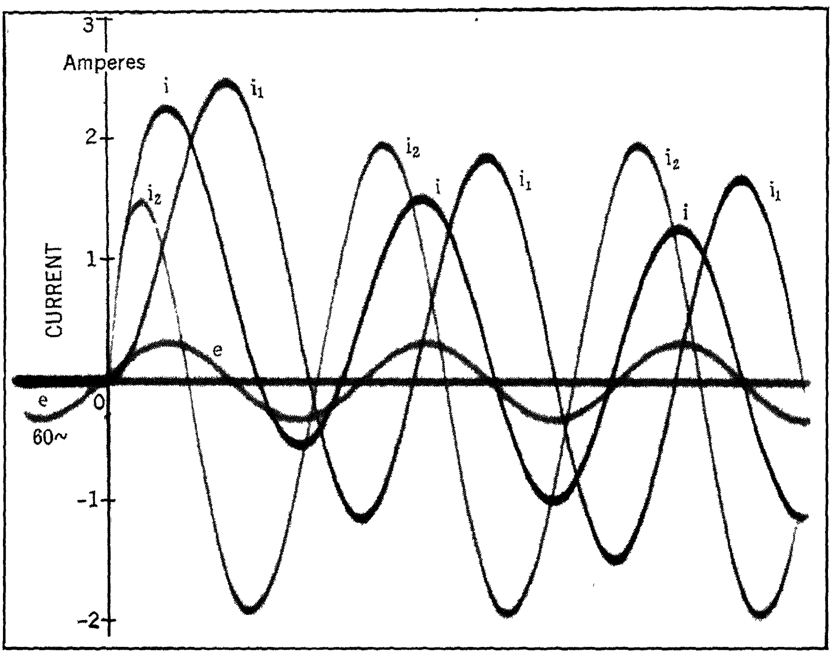
$$i_2 = \frac{24.5\sqrt{2}}{17.2}\sin(377t + \lambda + 66.°4) + \frac{24.5\sqrt{2}}{6.9}(-0.916)\cos(\lambda + 66.°4)\epsilon^{-862t} \tag{16}$$

When $\lambda = 0$:

$$i_1 = 1.62 \sin(377t - 80.°6) + 1.60\epsilon^{-62.5t} \tag{17}$$

$$i_2 = 2.02 \sin(377t + 66.°4) - 1.84\epsilon^{-862t} \tag{18}$$

Oscillogram 1 shows the graphs of i_1, i_2, and i for the particular case under discussion It is evident from the oscillographic records that the transient component of i_1 is a



OSCILLOGRAM 1.

Transient currents in parallel *RL* and *RC* branches.

e = 60-cycle sinusoidal applied emf. 24.5 volts, effective.
i_1 = transient current in the R_1L_1 branch.
i_2 = transient current in the R_2C_2 branch.
i = the sum of i_1 and i_2.
R_1 = 3.5 ohms. L_1 = 0.056 henry. R_2 = 6.9 ohms. C_2 = 168μf. $\lambda = 0°$.

positive d-c component while for i_2 it is a negative d-c component. The positive transient in i_1 is damped out at a much slower rate than the negative transient in i_2. The predominance of the positive d-c component in i is indicated on the oscillogram.

Oscillogram 2 illustrates the effect of closing the switch at the time of maximum positive voltage, namely, $\lambda = 90$ degrees.

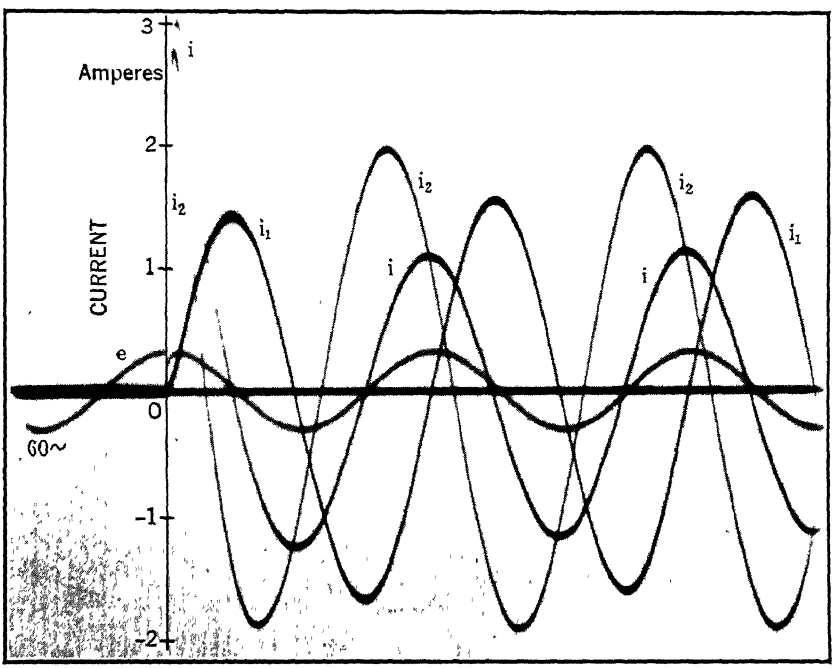
When $\lambda = 90°$:

$$i_1 = 1.62 \sin (377t + 9.°4) - 0.266\epsilon^{-62.5t} \tag{19}$$

$$i_2 = 2.02 \sin (377t + 156.°4) + 4.22\epsilon^{-862t} \tag{20}$$

Equation (19) indicates that the d-c component in the inductive branch is under these conditions much reduced and negative in sign. The magnitude of the sudden inrush of current to the capacitive branch is plainly predicted by equation (20).

R_1L_1 in Parallel with R_2L_2.—It is assumed that, beginning at $t = 0$, a voltage (e) is applied, for a finite length of time, to the R_1L_1 and the R_2L_2 branches shown in Fig. 2. So far as it concerns the present discussion, e may be any type of voltage, i.e., alternating, direct, or transient. If the source of voltage, e, is other than a momentary surge



OSCILLOGRAM 2.

Similar to Oscillogram 1 except for the time of closing the switch. $\lambda = 90°$ in this case as compared with $\lambda = 0°$ in Oscillogram 1.

(a condenser discharge, for example), it is assumed that the switch in Fig. 2 is opened at some later time.

The total charge that passes from the source to the branches during the interval $t = 0$ to $t = t_1$ is:

$$Q = \int_0^{t_1} i dt \tag{21}$$

This total charge will divide between the two branches inversely pro-

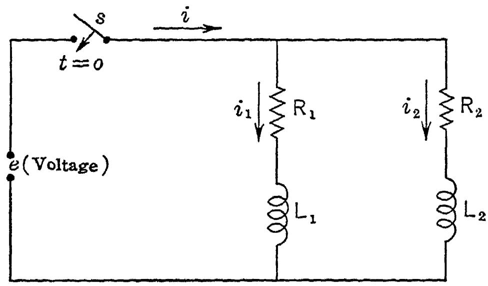


FIG. 2.—Inductive parallel branches to which a voltage, e, is applied at $t = 0$.

portional to the resistances of the individual branches. The following steps will serve to illustrate why the total charge divides as it does.

$$R_1 i_1 + L_1 \frac{di_1}{dt} = e \tag{22}$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} = e \tag{23}$$

or

$$R_1 i_1 + L_1 \frac{di_1}{dt} = R_2 i_2 + L_2 \frac{di_2}{dt} \tag{24}$$

$$\int_0^{t_1} R_1 i_1 \, dt + \int_0^{t_1} L_1 \frac{di_1}{dt} \, dt = \int_0^{t_1} R_2 i_2 \, dt + \int_0^{t_1} L_2 \frac{di_2}{dt} \, dt \tag{25}$$

If $i_1 = 0$ at $t = 0$ and $i_1 = 0$ at $t = t_1$ it follows that:

$$\int_0^{t_1} L_1 \, di_1 = 0$$

Similarly, if $i_2 = 0$ at $t = 0$ and $i_2 = 0$ at $t = t_1$:

$$\int_0^{t_1} L_2 \, di_2 = 0$$

Equation (25) reduces, therefore, to the following form:

$$R_1 \int_0^{t_1} i_1\, dt = R_2 \int_0^{t_1} i_2 dt \tag{26}$$

or

$$R_1 Q_1 = R_2 Q_2$$

from which

$$\frac{Q_1}{Q_2} = \frac{R_2}{R_1} \tag{27}$$

This fact makes it possible to employ a diverting shunt in connection with ballistic galvanometers which are used to measure quantities of electricity. The total charge that comes to the shunt and the galvanometer divides inversely proportional to their resistances. The self-inductances of shunt and galvanometer do not influence the division of the *total* charge.

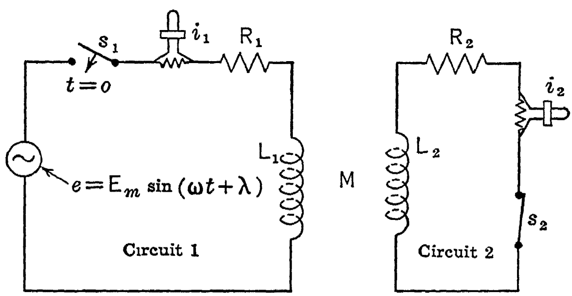


FIG. 3.—Inductively coupled circuits employed in obtaining Oscillograms 3 and 4.

INDUCTIVELY COUPLED CIRCUITS

With the circuit arrangement shown in Fig. 3, the conditions for dynamic equilibrium are:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = E_m \sin(\omega t + \lambda) \tag{28}$$

and

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = 0 \tag{29}$$

The Operational Solution.—The above equations take the following operational forms:

$$R_1 i_1 + L_1 i_1 p + M i_2 p = E_m \sin(\omega t + \lambda) \tag{30}$$

and

$$R_2 i_2 + L_2 i_2 p + M i_1 p = 0 \tag{31}$$

Solving (31) for i_2:

$$i_2 = \frac{-Mi_1p}{R_2 + L_2p} \tag{32}$$

Substituting this in (30):

$$R_1i_1 + L_1i_1p + Mp\frac{(-Mi_1p)}{R_2 + L_2p} = E_m \sin(\omega t + \lambda) \tag{33}$$

Solving explicitly for i_1:

$$i_1 = \frac{E_m \sin(\omega t + \lambda)\dfrac{(R_2 + L_2p)}{(L_1L_2 - M^2)}}{p^2 + \left(\dfrac{L_1R_2 + L_2R_1}{L_1L_2 - M^2}\right)p + \dfrac{R_1R_2}{(L_1L_2 - M^2)}} \tag{34}$$

A corresponding solution for i_2 yields:

$$i_2 = \frac{E_m \sin(\omega t + \lambda)\dfrac{(-Mp)}{(L_1L_2 - M^2)}}{p^2 + \dfrac{(L_1R_2 + L_2R_1)}{(L_1L_2 - M^2)}p + \dfrac{R_1R_2}{(L_1L_2 - M^2)}} \tag{35}$$

Having expressed the currents as:

$$i = E_m\frac{Y(p)}{Z(p)}\sin(\omega t + \lambda) \tag{36}$$

Heaviside's expansion theorem as applied to alternating potentials may be employed to effect the solutions in particular cases. In terms of generalized parameters the expressions become so cumbersome that they are meaningless.

The Conventional Type of Solution Applied to a Particular Case.—The conventional type of solution is also somewhat awkward from an algebraic point of view if generalized expressions are employed for the circuit parameters. It is, however, a straightforward matter to obtain the current solutions in particular cases where the circuit parameters can be given numerical values. And a great deal can be learned about the general current solutions from the numerical solutions of a particular case, provided of course that the particular case does not contain too many singular conditions.

The following relations may be established by solving equations (28) and (29) simultaneously:

$$\frac{d^2i_1}{dt^2} + \frac{R_1L_2 + R_2L_1}{(L_1L_2 - M^2)}\frac{di_1}{dt} + \frac{R_1R_2}{(L_1L_2 - M^2)}i_1 = \frac{E_mZ_2}{(L_1L_2 - M^2)}\sin(\omega t + \lambda + \theta_2) \tag{37}$$

and

$$\frac{d^2i_2}{dt^2} + \frac{R_1L_2 + R_2L_1}{(L_1L_2 - M^2)}\frac{di_2}{dt} + \frac{R_1R_2}{(L_1L_2 - M^2)}i_2 = \frac{-\omega ME_m}{(L_1L_2 - M^2)}\cos(\omega t + \lambda) \tag{38}$$

In equation (37):

$$Z_2 = \sqrt{R_2{}^2 + \omega^2 L_2{}^2}$$

and

$$\theta_2 = \tan^{-1} \frac{\omega L_2}{R_2}$$

The following circuit parameters, boundary conditions, and applied voltage have been selected for the numerical solutions:

$R_1 = 6.3$ ohms.	$E_m = 40$ volts, 60 cycles.
$L_1 = 0.092$ henry.	$\lambda = 0$ degree.
$M = 0.0265$ henry.	$i_1 = 0$ at $t = 0$.
$R_2 = 1.5$ ohms.	$i_2 = 0$ at $t = 0$.
$L_2 = 0.012$ henry.	

The i_1 Solution.—Substituting numerical values into (37), the equation reduces to the following form:

$$\frac{d^2 i_1}{dt^2} + 531 \frac{di_1}{dt} + 23{,}507\, i_1 = 474{,}600 \sin (\omega t + 71.^\circ 7) \tag{39}$$

The general solution of a first degree, second order differential equation is given in the Appendix, page 272. The explicit expression for i is:

$$i = \frac{h}{2b}\left[\epsilon^{\alpha_1 t} \int \epsilon^{-\alpha_1 t}\, e dt - \epsilon^{\alpha_2 t} \int \epsilon^{-\alpha_2 t}\, e dt\right] + c_1 \epsilon^{\alpha_1 t} + c_2 \epsilon^{\alpha_2 t} \tag{40}$$

The various symbols carry the definitions given to them on the pages immediately preceding page 272. In the particular case being investigated:

$e = \sin (\omega t + 71^\circ.7)$	$\alpha_1 = -48.7$
$h = 474{,}600$	$\alpha_2 = -482.3$
$2b = 433.6$	

If the above numerical values are substituted into equation (40) and the indicated integrations performed, the expression for i_1 becomes:

$$i_1 = 2.04 \sin (\omega t - 49^\circ) + c_1 \epsilon^{-48.5t} + c_2 \epsilon^{-485t} \tag{41}$$

Since $i_1 = 0$ at $t = 0$:

$$c_1 + c_2 = 1.539 \tag{42}$$

Another relationship must be established between c_1 and c_2. Under the assumed boundary conditions it is evident, from equation (29), that:

$$\left(\frac{di_2}{dt}\right)_{t=0} = -\frac{M}{L_2}\left(\frac{di_1}{dt}\right)_{t=0}$$

If, now, the boundary conditions are imposed on equation (28) it follows that:

$$\left(\frac{di_1}{dt}\right)_{t=0} = \frac{E_m L_2}{L_1 L_2 - M^2} \sin \lambda = 0$$

The derivative of i_1 may be determined by differentiating equation (41). Equating the derivative thus found to zero, the following relationship may establish at the time $t = 0$:

$$c_1 + 9.90c_2 = 10.36 \tag{43}$$

Simultaneous solution of (42) and (43) shows that.

$$c_1 = .548 \quad \text{and} \quad c_2 = 0.991$$

Therefore:

$$i_1 = 2.04 \sin(\omega t - 49°) + .548\epsilon^{-48.7t} + 0.991\epsilon^{-482.3t} \tag{44}$$

The i_2 Solution.—If the numerical values previously given are substituted into equation (38), it takes the form shown below:

$$\frac{d^2i_2}{dt^2} + 531\frac{di_2}{dt} + 23{,}507\, i_2 = -\,994{,}080 \cos \omega t \tag{45}$$

from which:

$$i_2 = 4.28 \sin(\omega t - 210.°6) + c_3\epsilon^{\alpha_1 t} + c_4\epsilon^{\alpha_2 t} \tag{46}$$

The α's have the same values as given for i_1, and c_3 and c_4 may be determined in a manner similar to that outlined for the evaluation of c_1 and c_2. It is found that:

$$c_3 = 0.778 \quad \text{and} \quad c_4 = -2.96$$

Therefore:

$$i_2 = 4.28 \sin(\omega t - 210.°6) + 0.278\epsilon^{-48.7t} - 2.96\epsilon^{-482.3t} \tag{47}$$

General Interpretation of the Current Solutions.—The "uncoupled" expression for i_1 is:

$$i_1' = 1.14 \sin(\omega t - 79.°7) + 1.12\epsilon^{-68.5t}. \tag{48}$$

Comparing the above expression with equation (44) reveals the general effect of the coupling upon i_1. The steady-state component of the current becomes greater in magnitude as a result of the coupling, and its phase angle with respect to the applied voltage becomes less. The first effect is due to the lowering of the effective inductance of the circuit, although it should be recognized that the effective resistance of the circuit is increased as a result of the coupling. It may be shown, from theoretical conditions, that the "effective coupled" values of primary inductance and primary resistance are:

$$\left[L_1 - \frac{M^2\omega^2L_2}{L_2{}^2\omega^2 + R_2{}^2}\right],$$

and

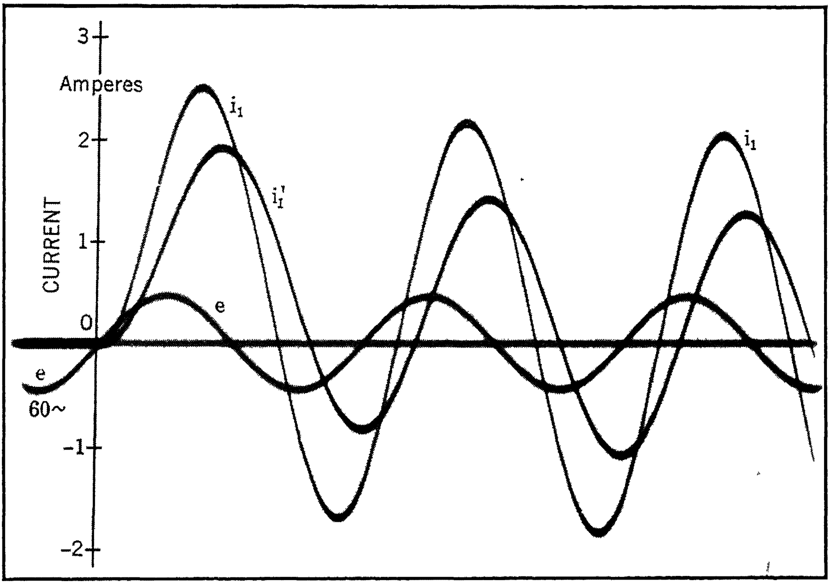
$$\left[R_1 + \frac{M^2\omega^2R_2}{L_2{}^2\omega^2 + R_2{}^2}\right]$$

In the particular case being considered the effective primary inductance changes from 0.092 henry (uncoupled) to 0.0392 henry (coupled). The resistance changes from 6.3 ohms (uncoupled) to 12.9 ohms (coupled).

Two exponential terms appear in the expression for i_1 when circuit 1 is coupled to circuit 2. One of these terms is damped out at a slower rate than the single

exponential term found in the " uncoupled " expression for i_1; the second is damped out at a much faster rate.

Equation (47) shows that i_2 is of the same general nature as i_1, containing as it does a steady-state term and two corresponding exponential transient terms. The damping factors in i_2 are similar to those in i_1. The magnitudes of the transient terms in i_2 are, of course, not equal to those in i_1. The transient terms combine with the steady-state terms in such a manner as to make the resultant currents in both circuits equal to zero at $t = 0$. The steady-state term in i_2 lags behind the primary voltage by 210.°6. The relative position of the two steady-state terms is given by equations (44) and (47). i_{2ss} lags i_{1ss} by 161.°6 in this particular case.



OSCILLOGRAM 3.

Illustrating the effect of mutual inductance upon i_1.

i_1' is the current in circuit 1 (Fig. 3) with s_2 open.

i_1 is the current in circuit 1 with s_2 closed.

e = 60-cycle sinusoidal applied emf. 28 volts, effective.

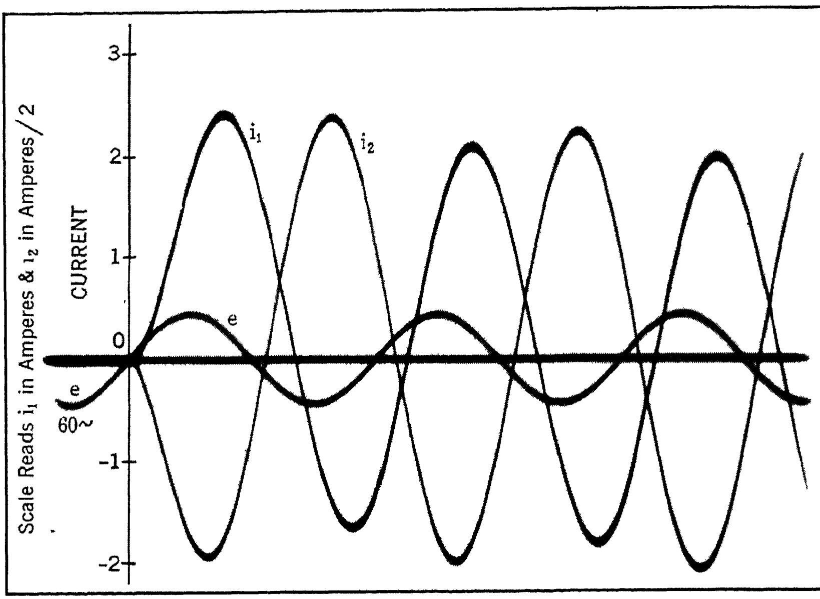
$R_1 = 6.3$ ohms. $L_1 = 0.092$ henry. $M = 0.026$ henry.

$R_2 = 1.5$ ohms. $L_2 = 0.012$ henry. $\lambda = 0°$.

Oscillographic Verification.—For a particular set of parameters the transient effects are most easily analyzed by means of oscillograms. One of the most interesting of the phenomena to be observed in a coupled circuit of this type is that the mutual inductance acts to decrease the effective self-inductance of circuit 1. Oscillogram 3 shows the current variation in circuit 1 both when circuit 2 is open and when it is closed. The smaller current flows in the primary when the secondary is open.

When the secondary circuit is closed the primary current increases in magnitude and changes its phase position so that it is more nearly in phase with the applied voltage. The increase in the steady-state component of the current as well as the change in phase position is shown on Oscillogram 3.

The relationship between the primary and secondary currents of Fig. 3 is shown in Oscillogram 4 for the case of $\lambda = 0°$. It is evident that, for the particular set of parameters employed and $\lambda = 0°$, the



OSCILLOGRAM 4.

Illustrating the relative phase positions of i_1 and i_2 throughout the transient period in the $R_1L_1M\ R_2L_2$ combination.

i_1 is the current in circuit 1. i_2 is the current in circuit 2.

Circuit parameters similar to those given in connection with Oscillogram 3.

transient components are not large as compared with the steady-state components. The instantaneous values of i_1 and i_2 during the first half cycle do not differ materially from steady-state values. The relative phase positions of the two currents are shown throughout the transient period, and the fact that the steady-state secondary current lags the steady-state primary current by less than 180 degrees is clearly illustrated.

SYSTEMATIZING THE GENERAL PROCEDURE

The methods which have been employed thus far do not lend themselves readily to the solution of current in the case of a general network. By a general network is meant an electrical system having any number of meshes. A mesh is one complete loop about which Kirchhoff's emf law may be applied. In a general case the parameters of the various meshes will be composed of the parameters of the individual branches which comprise the mesh. A branch is considered to be the series element between any two junction points of the network. Any or all branches, in the general network may possess R, L, C, and M.

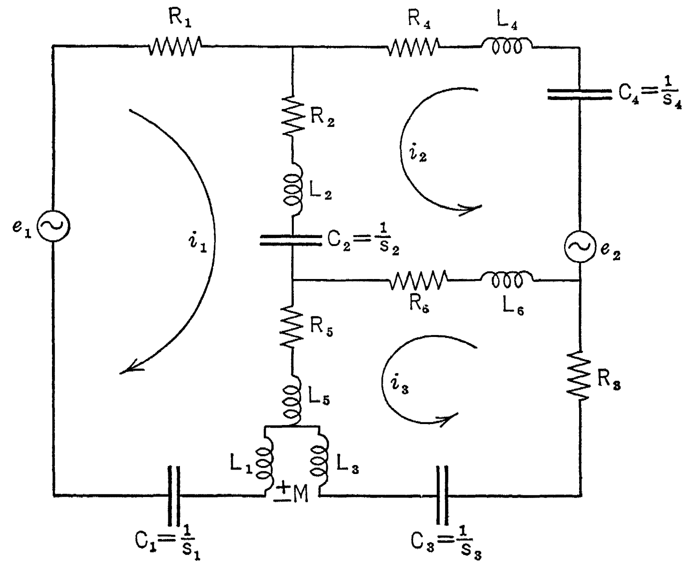


FIG. 4.—Illustrating the use of "mesh" currents in a network having six "branches."

In the direct application of Kirchhoff's laws, the current in each branch is treated as a distinct dependent variable. Reference to Fig. 4 will show that there are six branches in this particular network. The number of dependent variables may at once be reduced by considering mesh currents rather than branch currents. In the case shown in Fig. 4, the three mesh currents, i_1, i_2, and i_3, replace the six individual branch currents. The same reduction in the number of dependent variables can, of course, be accomplished by the application of Kirchhoff's current law after the circuit equations have been formulated in terms of branch currents. But the practice of using mesh currents accomplishes the same purpose and at the same time simplifies the

procedure of writing down the circuit equations for the more complicated networks. It is evident that, if the mesh currents can be found, the current in each individual branch can easily be determined.

An attempt to write Kirchhoff's emf law around the three loops or meshes indicated in Fig. 4 using ordinary notation results in very long equations. So many terms are involved that the likelihood of errors creeping in is great. The following device is employed to simplify the emf equations.

All the resistance encountered by the current, i_1, is called R_{11}. It is the resistance of mesh 1 to current 1. In the particular case shown in Fig. 4:

$$R_{11} = (R_1 + R_2 + R_5) \tag{49}$$

All the self-inductance of mesh 1 to current 1 is:

$$L_{11} = (L_1 + L_2 + L_5) \tag{50}$$

All the series capacitance of mesh 1 to current 1 is:

$$C_{11} = \frac{1}{1/C_1 + 1/C_2} \tag{51}$$

It is somewhat more convenient to express the series capacitance effect in terms of the reciprocal of capacitance, namely, elastance. If this is done the elastance of mesh 1 to current 1 becomes:

$$S_{11} = S_1 + S_2 \tag{52}$$

But in addition to the counter-voltages established in mesh 1 by i_1, counter-voltages are established in mesh 1 by i_2 and i_3. It is convenient to designate the common elements between mesh 1 and mesh 2 as follows:

$$\left.\begin{aligned} R_{12} &= R_{21} = R_2 \\ L_{12} &= L_{21} = L_2 \\ S_{12} &= S_{21} = S_2 \end{aligned}\right\} \text{for network shown in Fig. 4.}$$

R_{12} means the resistance in mesh 1 which carries current 2. R_{21} means the resistance in mesh 2 which carries current 1. In general these quantities are equal, and in the particular case under discussion they are equal to R_2.

Likewise, L_{12} means the portion of the inductance of mesh 1 which is encountered by current 2. L_{21} means the inductance of mesh 2 which is encountered by current 1.

Similar meanings are attached to S_{12} and S_{21}.

It should be noted that the common inductance between meshes

must include any mutual inductive effects that exist between the two meshes. For example, the total inductance of mesh 1 to current 3 is:

$$L_{13} = L_5 \pm M \tag{53}$$

The sign of M is determined by the mode of winding the L_1 and L_3 coils and the space positions of these windings in relation one to the other. If the above conditions are such that $L_5 \dfrac{di_3}{dt}$ induces a counter-voltage in mesh 1 in the same direction as does $M \dfrac{di_3}{dt}$, then M is positive and adds directly to L_5 to form L_{13}. A more complete discussion of the sign of M may be found in Chapter V. The arbitrarily assumed directions of the currents will ultimately determine the signs of the counter-voltages in the mathematical expressions. For example, the entire counter-voltage $(L_5 + M) \dfrac{di_3}{dt}$ which acts in mesh 1 would be negative if the assumed positive direction of i_3 were reversed from that shown in Fig. 4.

The inductive counter-voltages in mesh 3 due to $\dfrac{di_1}{dt}$ may be represented by $(L_5 \pm M) \dfrac{di_1}{dt}$, and the inductance of mesh 3 to current 1 is designated as follows:

$$L_{31} = (L_5 \pm M) \tag{54}$$

Kirchhoff's emf equations may now be written in a much neater form than would be the case if each individual circuit parameter were considered. The emf equations for the meshes of Fig. 4 take the following form:

Mesh 1:

$$\left(R_{11} + L_{11}\frac{d}{dt} + S_{11}\int dt\right) i_1 + \left(R_{12} + L_{12}\frac{d}{dt} + S_{12}\int dt\right) i_2$$
$$+ \left(R_{13} + L_{13}\frac{d}{dt} + S_{13}\int dt\right) i_3 = e_1 \tag{55}$$

Mesh 2:

$$\left(R_{21} + L_{21}\frac{d}{dt} + S_{21}\int dt\right) i_1 + \left(R_{22} + L_{22}\frac{d}{dt} + S_{22}\int dt\right) i_2$$
$$+ \left(R_{23} + L_{23}\frac{d}{dt} + S_{23}\int dt\right) i_3 = e_2 \tag{56}$$

Mesh 3:

$$\left(R_{31} + L_{31}\frac{d}{dt} + S_{31}\int dt\right) i_1 + \left(R_{32} + L_{32}\frac{d}{dt} + S_{32}\int \frac{d}{dt}\right) i_2$$

$$+ \left(R_{33} + L_{33}\frac{d}{dt} + S_{33}\int dt\right) i_3 = 0 \qquad (57)$$

The signs of i_1, i_2, and i_3 will be determined in each of the above equations by the arbitrarily assumed directions which are placed on the working diagram of the circuit. The principal mesh current in each of the above equations is considered to be positive when it flows in the arrow direction. All other currents are then positive in this particular mesh equation if their arrow directions in the common branches coincide with the arrow direction of the principal current. The principal current in mesh 1 is current 1; in mesh 2, current 2, etc. For the sake of illustration consider mesh 1. i_1, i_2, and i_3 are all positive in equation (55) because their arrow directions coincide in the common branches.

However, in equation (56), which states the condition for equilibrium in mesh 2, i_2 and i_3 have opposite arrow directions in the common branch R_6L_6. It follows that i_3 must be treated as a negative quantity in equation (56) if the values of R, L, and C are considered to be inherently positive. This fact becomes evident when it is recognized that a positive R_6i_3, for example, is opposite in direction to a positive R_6i_2.

In equation (57) i_3 is considered positive in order that the major counter-voltages $R_{33}i_3$, $L_{33}\frac{di_3}{dt}$, and $S_{33}\int i_3 dt$ shall be positive. In accordance with the conventions previously discussed i_1 will be positive in equation (57) since it acts in the common branch, R_5L_5M, in the same direction as positive i_3. i_2 is treated as a negative quantity in equation (57) since its arrow direction is opposite to the arrow direction of i_3 in their common branch R_6L_6.

The forms of equation (55), (56), and (57) immediately suggest a further simplification in notation. The simplification referred to is to let:

$$\left(R_{11} + L_{11}\frac{d}{dt} + S_{11}\int dt\right) = z_{11}$$

$$\left(R_{22} + L_{22}\frac{d}{dt} + S_{22}\int dt\right) = z_{22}$$

etc.

and:

$$\left(R_{12} + L_{12}\frac{d}{dt} + S_{12}\int dt\right) = z_{12}$$

$$\left(R_{21} + L_{21}\frac{d}{dt} + S_{21}\int dt\right) = z_{21}$$

etc.

The general network problem may now be formulated since the n simultaneous differential equations which describe the performance of an n-mesh network may be written as follows:

$$\left.\begin{array}{l} z_{11}\, i_1 + z_{12}\, i_2 + \ .\ .\ .\ + z_{1n}\, i_n = f_1(t) \\ z_{21}\, i_1 + z_{22}\, i_2 + \ .\ .\ .\ + z_{2n}\, i_n = f_2(t) \\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ z_{n1}\, i_1 + z_{n2}\, i_2 + \ .\ .\ .\ + z_{nn}\, i_n = f_n(t) \end{array}\right\} \tag{58}$$

z_{xy} is the generalized impedance of the xth mesh to the yth current. For example:

$$z_{xy} = \left(R_{xy} + L_{xy}\frac{d}{dt} + S_{xy}\int dt\right)$$

The values of R, L, and S are assumed to be constant.

The right-hand members of the above equations represent the driving emfs of the various meshes, which in general cases are functions of time. In the usual type of network, however, certain of the $f(t)$'s are zero, and with a single source of emf the equations may be so written that all $f(t)$'s except one are equal to zero.

The system of simultaneous equations represented by (58) is a most concise statement of the general network problem. It systematizes the establishment of the equilibrium equations and in particular cases it provides a ready means for determining the advisability of mathematical predetermination. In the simple circuit configurations which have been treated in this text, the simultaneous solution of the two or three differential equations which have been involved has presented no particular difficulties. If, however, only one or two additional meshes are added to some of these simple configurations, ordinary methods of solution will become wholly inadequate. Under these conditions the problem can, at least, be formulated as outlined in (58). If a solution by ordinary methods appears to be inadvisable one of the more advanced methods must be resorted to if a solution is to be effected. Among the more powerful of the advanced methods are:

(*a*) the determinant method of solution,[1] and (*b*) the "extended" Heaviside method.[2]

EXERCISES

1. Graph equations (17), (18), (19), and (20). Compare the resultant graphs with the records contained in Oscillograms 1 and 2.

2. Carry out all the details connected with the developments of equations (44) and (47).

3. Graph the component parts of equations (44) and (47) and compare the resultant graphs with the records shown in Oscillogram 4.

4. Write the fundamental equations for equilibrium in the transformer circuits (Fig. 3) employing the ordinary concepts of primary leakage flux, mutual flux, and secondary leakage flux. Define each term that is used Assign reasonable numerical values to the various component fluxes, number of turns, etc., and solve the fundamental equations for the particular case involved. Interpret the results.

5. Select, arbitrarily, a simple type of network in which to try out the " mesh " current method of solution. Write the fundamental equations in the form shown in (58). Assign, at pleasure, numerical values to the circuit parameters. After selecting a particular driving voltage (or voltages) solve the fundamental equations.

[1] Guillemin, "Communication Networks," Vol. I.

[2] Carson, "Electric Circuit Theory and the Operational Calculus."

CHAPTER X

VARIABLE CIRCUIT PARAMETERS

In many of the circuits employed in actual practice the parameters are not constant. R, L, and C may be either directly or indirectly functions of current, voltage, or time. For example, R most generally varies with temperature and in so doing becomes a function of both current and time. Another very common example is the variation of L in iron-clad circuits wherein L becomes a function of the magnetic saturation and therefore a function of the magnetizing current. Mathematical predetermination of the transient effects under these conditions requires that the exact nature of the particular variation be known. Unfortunately the problems do not admit of generalized solutions. Each particular case must be analyzed separately, and very often it becomes necessary to employ a step-by-step method because of mathematical difficulties.

VARIABLE RESISTANCE

Physical Considerations.—Except for the so-called zero temperature coefficient alloys, all metallic conductors have positive temperature coefficients. Between the limits of 0° and 100° C most pure metals have about the same temperature coefficient and within this range the resistance may be written to a fair degree of accuracy as a simple function of temperature in the well-known form:

$$R_T = R_0(1 + \alpha_0 T)$$

where R_0 is the resistance at 0° C.

$$\alpha_0 = 0.0042, \text{ approximately, for pure metals.}$$

T is the change in temperature in degrees centigrade from zero. A greater range of temperature may be covered by employing additional terms, for example:

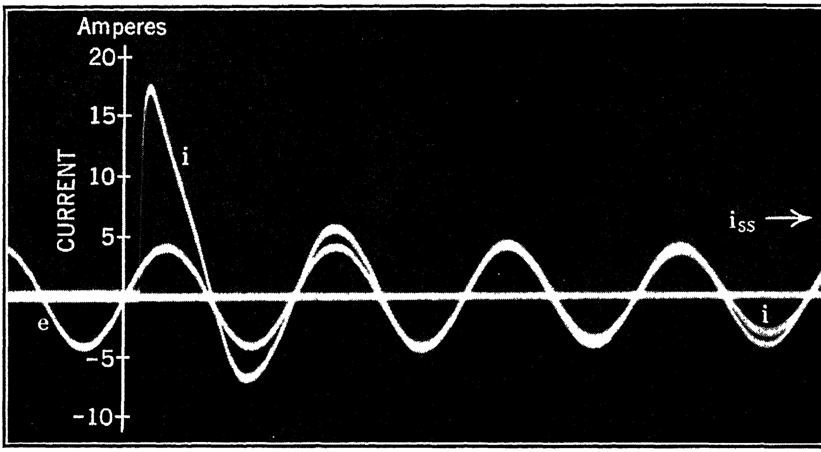
$$R_T = R_0(1 + \alpha_0 T + \beta_0 T^2 + \gamma_0 T^3)$$

β_0 and γ_0 are empirical constants and for a particular metal may be determined from an experimental graph of resistance versus temperature.

Most of the solids that are classed as insulators, practically all

solutions, and carbon have negative temperature coefficients. The resistance of these substances decreases with an increase in temperature. In a particular case the resistance may be written as a function of temperature in a manner similar to that described above.

But the fact that the resistance may be expressed as a function of temperature is of little value in transient predetermination. Since the variation in resistance is inherently due to change in temperature it becomes a complicated function of both current and time.



OSCILLOGRAM 1.

Momentary inrush of current to a 150-watt tungsten-filament lamp.

e = 60-cycle, applied emf. E (eff.) = 116 volts.

i = current taken by lamp.

Maximum instantaneous value of initial current = 17.5 amp.

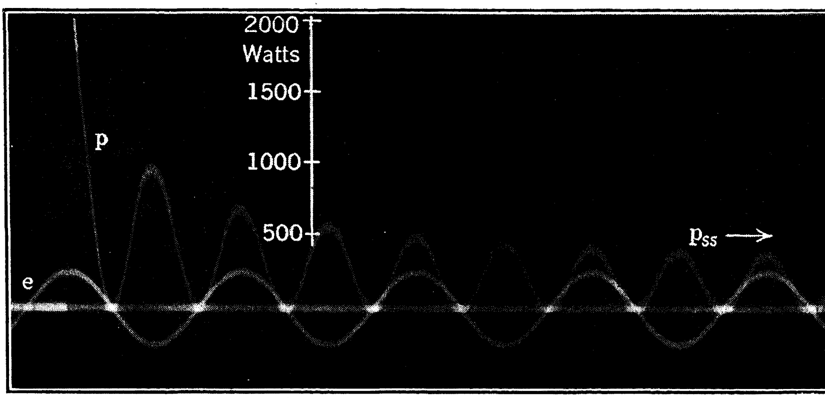
Maximum value of steady-state current = 1.84 amp.

Final steady-state current is not shown.

Oscillographic Illustrations.—In the case of tungsten-filament incandescent lamps, the change in resistance from room temperature to operating temperature is very great with the result that a marked increase in resistance occurs when rated voltage is suddenly applied to the filament. Oscillogram 1 illustrates the effect of such a variation in resistance upon the starting current. In this particular case the resistance changes from 8 ohms, at $t = 0$, to 89 ohms under steady operating conditions. It will be observed that the greatest change occurs during the first one-hundredth of a second. Ultimate steady-state values are not shown on the oscillogram because the remaining period of transition is relatively long and the actual change during the later period is not

great. The oscillogram shows that the circuit was closed at about the 30-degree point on the voltage wave. A considerably larger initial inrush of current would have resulted if the circuit had been closed at the point of maximum voltage.

The graph of instantaneous power in a variable resistance circuit is shown in Oscillogram 2. In this case the circuit is completed at the time of maximum voltage and the maximum instantaneous value of the power is more than thirteen times as large as the average power taken by the circuit under steady-state conditions.



OSCILLOGRAM 2.

Momentary inrush of power to a 150-watt tungsten-filament lamp.

e = 60-cycle, applied emf. E (eff.) = 116 volts.

p = power delivered to the lamp.

Maximum instantaneous value of initial power = 2000 watts.

Maximum value of steady-state power = 300 watts.

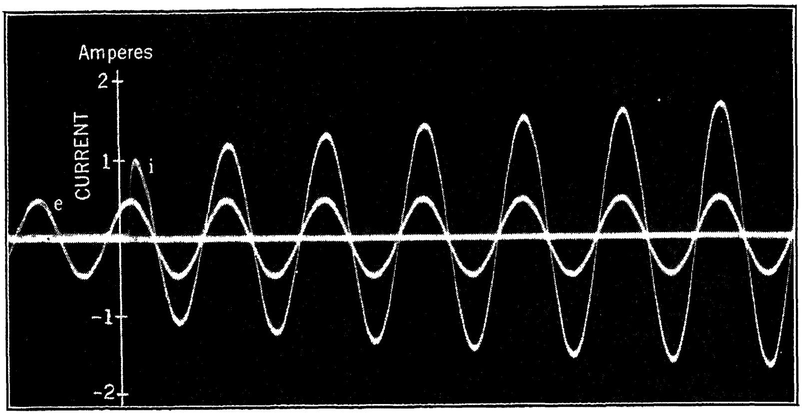
Final steady-state power is not shown.

The effect of the negative temperature coefficient of carbon is illustrated by Oscillograms 3 and 4. The gradual increase of the successive maxima of the current graph shown in Oscillogram 3 indicates that the resistance of the circuit is decreasing with the lapse of time. The initial resistance of the particular carbon circuit employed is 190 ohms as compared with the final steady-state value of 96.5 ohms. The high power factor of the circuit throughout the transient period is shown by the power graph in Oscillogram 4.

The transient phenomena shown by all the oscillograms in this article are of relatively short duration because of the low thermal inertia of the lamp filaments. Radical changes in resistance during short periods of time have been indicated. In certain other types

of circuits, notably the copper field windings of generators and motors, the change in resistance is not so pronounced but appreciable change may take place over the period of an hour or longer.

Effect of Variable Resistance under Steady-State Conditions.— With alternating potential difference applied, variable resistance may



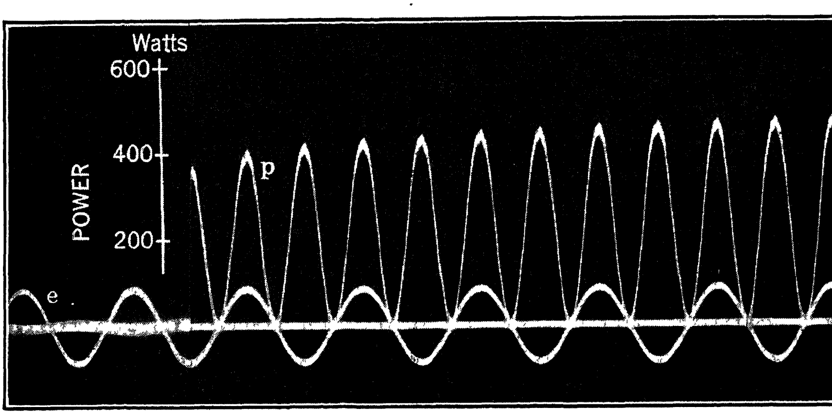
OSCILLOGRAM 3.

Illustrating the effect of the negative temperature coefficient of carbon lamps upon the starting current.

e = 60-cycle, applied emf. E (eff.) = 116 volts.

i = current taken by three 115-volt, 50-watt carbon-filament lamps in parallel.

Steady-state value of the current = 1.2 amp. (eff.).



OSCILLOGRAM 4.

Power delivered to carbon lamps.

e = 60-cycle, applied emf. E (eff.) = 115 volts.

p = power variation. (Five 115-volt, 50-watt, carbon-filament lamps in parallel.)

Average steady-state power = 250 watts.

continue to affect the performance of the circuit even under the so-called steady-state conditions. If the frequency of the applied emf is sufficiently low and the thermal inertia of the circuit is sufficiently small the resistance of the circuit will change periodically. Assuming that the material has a positive temperature coefficient, the resistance will periodically be low in the region of zero current and periodically high in the region of maximum current. If appreciable change in resistance occurs the current will acquire a flat-topped wave form even though a sinusoidal emf is applied. Broken up into its harmonic components such a wave form will be found to possess an appreciable third harmonic. Fig. 1 illustrates the manner in which a third harmonic current combines with its fundamental to give a flat-topped wave. The equation for such a curve may be written as follows:

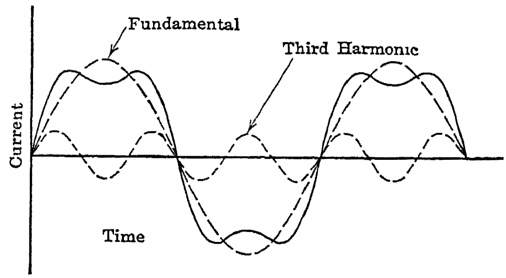


FIG. 1.—Illustrating the manner in which the fundamental and third harmonic components may be combined to form an approximately flat-topped wave form. Harmonics higher than the third are required if a flat wave form is to have its maximum at the quarter-cycle point.

$$i = I_{m1} \sin \omega t + I_{m3} \sin 3\omega t$$

In the actual current wave certain higher harmonics will also be present.

VARIABLE INDUCTANCE-CONSTANT POTENTIAL APPLIED

Physical Considerations.—The majority of the magnetic circuits employed in actual practice contain iron or steel. Under these conditions the magnetic flux produced by a given mmf is influenced by several factors, chief amongst which are: the magnetic saturation, the previous state of magnetization, and the particular grade of iron or steel used. Since self-inductance depends upon the flux-linkages $[N\phi]$ per unit current, L will vary whenever $N\phi/i$ is not a constant.

$$L = \frac{N\phi}{i} \text{ henrys} \tag{1}$$

where N is the number of turns.

ϕ is the magnetic flux in webers linking N turns.
i is the current in amperes.

Assuming that the flux, linking with the electric circuit of N turns, varies in accordance with the magnetization curve shown in Fig. 2, it is evident that $N\phi/i$ may vary between wide limits depending upon the degree of saturation. At ϕ equal to 0.01 weber, the self-inductance of a concentrated coil of 1000 turns as calculated by equation (1) is 20 henrys. At ϕ equal to 0.012 weber, the effective self-inductance is only 15 henrys. Obviously, such a change in L will greatly influence the transient current and related quantities.

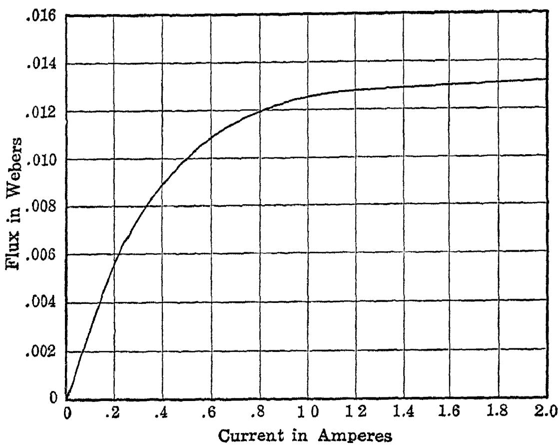


FIG. 2.—Magnetization curve for the magnetic circuit employed in the illustrative examples.

If hysteresis effects are neglected the magnetic flux may, to a reasonable degree of accuracy, be represented by Frölich's equation.

$$\phi = \frac{mi}{1 + ni} \text{ webers} \tag{2}$$

where m and n are empirically determined constants for the particular magnetic circuit.

ϕ is the flux produced by the exciting current i.

L may, therefore, be expressed as a function of the current by substituting for ϕ in equation (1), thus:

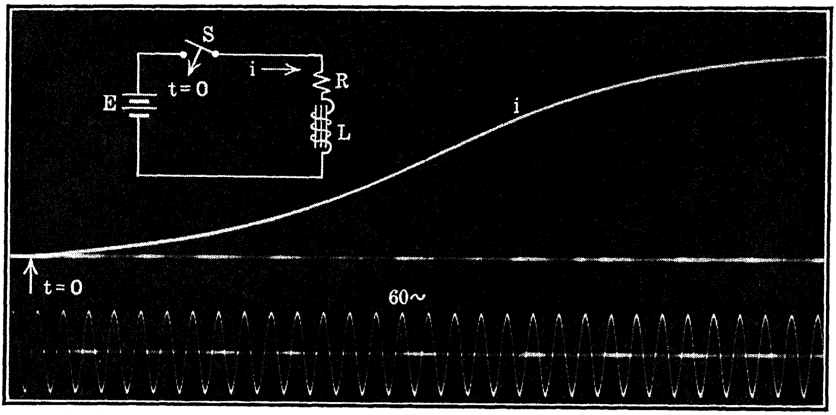
$$L = \frac{Nm}{1 + ni} \text{ henrys} \tag{1}$$

The above expression shows how L decreases as the magnetizing current increases.

If a constant potential difference is suddenly applied to an iron-clad circuit the current, in general, will build up at a relatively slow rate near $t = 0$ because of the high inductance. Assuming that the resistance is constant the rate of building up will increase as magnetic saturation is approached due to the reduction in the value of L. Oscillo-

gram 5 illustrates the nature of the current variation with respect to time in a variable inductance circuit.

The Use of Frölich's Equation.—When the variation of ϕ can be expressed as a simple function of the magnetizing current the relation-



Oscillogram 5.

Illustrating the nature of the growth of current in a variable inductance circuit.

ship between current and time may be established mathematically [1] as follows:

The basic relationship is:

$$Ri + \frac{d}{dt}(Li) = E \tag{3}$$

or

$$Ri + L\frac{di}{dt} + i\frac{dL}{dt} = E \tag{4}$$

Substituting for the value of L:

$$Ri + \frac{Nm}{1+ni}\frac{di}{dt} - \frac{Nmni}{(1+ni)^2}\frac{di}{dt} = E \tag{5}$$

Separating variables:

$$\left[\frac{Nm}{1+ni} - \frac{Nmni}{(1+ni)^2}\right]\frac{di}{dt} = E - Ri$$

$$\left[\frac{Nm(1+ni) - Nmni}{(1+ni)^2(E-Ri)}\right]di = dt$$

$$\frac{di}{(E-Ri)(1+ni)^2} = \frac{dt}{Nm} \tag{6}$$

[1] Berg, "Electrical Engineering—Advanced Course," Chapter III.

from which:
$$\frac{t}{Nm} = \int \frac{di}{(E - Ri)(1 + ni)^2} + c_1 \tag{7}$$

The integration is most easily effected by separating the denominator of the integral into partial fractions.

$$\frac{1}{(E - Ri)(1 + ni)^2} = \frac{A}{(E - Ri)} + \frac{B}{(1 + ni)} + \frac{D}{(1 + ni)^2} \tag{8}$$

where A, B, and D are constants that may be evaluated. A conventional method of evaluating these constants is to recognize that equation (8) must be satisfied for any and all values of i. Clearing fractions in equation (8), yields:

$$1 = A(1 + ni)^2 + B(E - Ri)(1 + ni) + D(E - Ri)$$

when $i = E/R$:

$$A = \frac{1}{\left(1 + \frac{nE}{R}\right)^2} = \frac{R^2}{(R + nE)^2}$$

when $i = 0$:

$$B = \frac{1 - A - DE}{E} = \frac{Rn}{(R + nE)^2}$$

when $i = -1/n$:

$$D = \frac{1}{\left(E + \frac{R}{n}\right)} = \frac{n}{R + nE}$$

Equation (7) may now be written:

$$\frac{t}{Nm} = \int \frac{A\,di}{E - Ri} + \int \frac{B\,di}{(1 + ni)} + \int \frac{D\,di}{(1 + ni)^2} + c_1$$

Integration yields:

$$\frac{t}{Nm} = -\frac{A}{R} \log (E - Ri) + \frac{B}{n} \log (1 + ni) - \frac{D}{n} \frac{1}{(1 + ni)} + c_1 \tag{9}$$

The constant of integration, c_1, may be evaluated from the boundary condition, namely, that $i = 0$ at $t = 0$. Thus:

$$0 = -\frac{A}{R} \log E - \frac{D}{n} + c_1$$

from which:

$$c_1 = \frac{A}{R} \log E + \frac{D}{n}$$

By proper manipulation equation (9) becomes:

$$t = \frac{Nm}{(R + nE)}\left[\frac{R}{(R + nE)} \log E \frac{(1 + ni)}{(E - Ri)} + \frac{ni}{1 + ni}\right] \tag{10}$$

Equation (10) states the mathematical relationship that exists between current and time, and therefore a graph of i versus t may be obtained by assuming various values of i and solving for the corresponding values of t. The time required for the establishment of a given current, i, may be calculated directly.

The Method of Finite Differences.—Aside from oscillographic solutions, the most practical method of solving transient problems involving variable circuit parameters is, in general, the method of finite differences. The calculations are based on the assumption that the parameters remain constant over a small finite interval of time. The predetermination of the transient current in an iron-clad RL circuit with constant potential difference applied will serve to illustrate the details connected with the method. It will be assumed that the flux varies in accordance with the magnetization curve[2] shown in Fig. 2, and that 12 volts are applied at $t = 0$. The effective number of turns is assumed to be 1000, the resistance of which is 10.6 ohms. The emf equation is:

$$Ri + N\frac{d\phi}{dt} = E$$

where ϕ is in webers if practical units of R, i, and E are employed.

Substituting numerical coefficients and finite differences of ϕ and t changes the equation to:

$$10.6i + 1000\frac{\Delta\phi}{\Delta t} = 12$$

Since $i = 0$ at $t = 0$:

$$1000\frac{\Delta\phi_1}{\Delta t_1} = 12, \text{ as a first approximation}$$

Judgment must be exercised in the arbitrary selection of the interval Δt_1. In this particular case it should be of the order of one-tenth of the estimated time constant of the circuit. A rough approximation of the time constant may be obtained by estimating an average value of L from the magnetization curve. Estimating L to be 12 henrys, a rough approximation of the time constant is 1 second. Therefore Δt_1 is selected as 0.1 second.

[2] The effects of the eddy currents in producing demagnetization and non-uniform space distribution of flux are neglected. These factors are very important when parts of the magnetic circuit are solid iron. See "Transients in Magnetic Systems" by C. F. Wagner in *Electrical Engineering*, March, 1934.

Neglecting the Ri drop during the first interval of time, the change in flux during Δt_1 is:

$$\Delta\phi_1 = \frac{12(0.1)}{1000} = 0.0012 \text{ weber}$$

From the magnetization curve it is found that $\Delta\phi_1$ requires, for its establishment, 0.041 ampere. This value of current is assumed to flow in the circuit during the second interval of time, Δt_2. It follows that:

$$1000\frac{\Delta\phi_2}{\Delta t_2} = 12 - 10.6(0.041)$$

or

$$\Delta\phi_2 = \frac{0.1}{1000}(12 - 0.435) = 0.001156 \text{ weber}$$

The flux established in the magnetic circuit at the end of the second interval of time is, under the present assumptions:

$$\Sigma\Delta\phi = \Delta\phi_1 + \Delta\phi_2 = 0.002356 \text{ weber}$$

$\Sigma\Delta\phi$ indicates the value of current to be used in making succeeding calculations. The numerical results of the step-by-step method are:

$E = 12$ Volts

Step	t Seconds	$\Delta\phi$ Webers	$\Sigma\Delta\phi$ Webers	i Amperes
1	0	0	0	0
2	0 1	0 00120	0012	0 041
3	0.2	00115	00235	0 083
4	0.3	.00111	00346	0 120
5	0 4	00107	.00454	0 160
6	0 5	.00103	.00557	0 200
7	0 6	00098	.00655	0 245
8	0 7	.00094	00745	0.300
9	0 8	.00088	.00838	0.360
10	0 9	.00081	.00919	0 425
11	1.0	00074	00994	0.500
12	1.1	00067	.01061	0 580
13	1.2	00058	.01120	0 660
14	1 3	00050	.01170	0 770
15	1 4	00038	.01208	0 850
16	1.5	.00029	01238	0.940
17	1.6	.00020	.01259	1.040
18	1.7	.00010	.01268	1 110
19	∞	.00000	01270	1 133

Note: The accuracy of method does not warrant writing any single numerical value to four significant figures.

Assuming that 15 volts are applied to the same circuit at $t = 0$, the following table of values may be determined by the step-by-step method.

$E = 15$ Volts

Step	t Seconds	$\Delta\phi$ Webers	$\Sigma\Delta\phi$ Webers	i Amperes
1	0	0	0	0
2	0 1	0 00150	0 00150	0 055
3	0 2	00144	00294	0.105
4	0 3	00138	00433	0.152
5	0 4	00133	00567	0 205
6	0 5	00128	00695	0 270
7	0.6	00121	00816	0 345
8	0 7	00111	.00928	0 430
9	0 8	00104	.01032	0 540
10	0 9	.00092	01125	0 670
11	1 0	00079	.01204	0 840
12	1 1	00061	01265	1 080
13	∞	0	01300	1 410

The graphs of the currents versus time for the two values of applied voltage are shown in Fig. 3. The effect of the higher voltage is to decrease

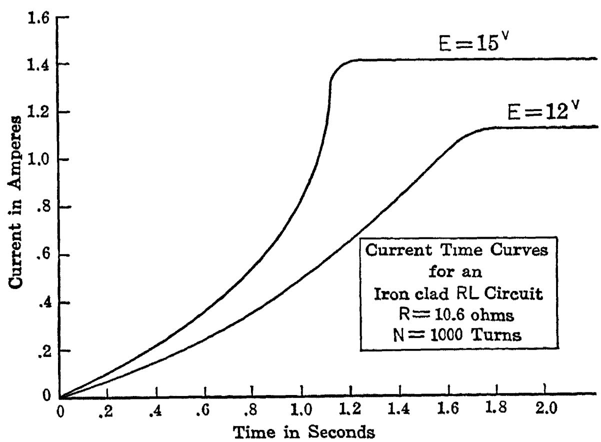


FIG. 3.—The growth of current in a variable inductance circuit as calculated by the method of finite differences.

the time required for the current to reach its E/R value. The higher voltage applied to the circuit causes a larger steady-state current to

flow and this larger current, in turn, sets up a larger total flux which makes the value of L very small during the last part of the transient period. The decrease in L is evident from the rapid rise in current near the end of the transient period. Actually the transition of current in this region is much smoother than is indicated by the step-by-step method of calculation.

If increased accuracy is desired succeeding approximations may be made. As a second approximation the Ri drop during each interval of time may be determined on the basis of the average value of the current calculated in the first approximation. Various other refinements may be employed in connection with the method of finite differences.

ALTERNATING POTENTIAL DIFFERENCE APPLIED TO A VARIABLE INDUCTANCE CIRCUIT

The method of finite differences described above may be used to predict the transient current in a variable inductance circuit to which an alternating potential difference is suddenly applied. The basic relationship that must be satisfied is:

$$Ri + N\frac{d\phi}{dt} = E_m \sin(\omega t + \lambda) \tag{11}$$

where ϕ is in webers and the other quantities are expressed in the practical system of units.

Employing finite differences:

$$Ri + N\frac{\Delta\phi}{\Delta t} = E_m \sin(\Sigma\Delta\theta + \lambda) \tag{12}$$

where $\Sigma\Delta\theta = \Sigma\Delta\omega t$, the angular measure of the point on the voltage wave which is under investigation.

Illustrative Example.—In any particular case the selection of the size of Δt is governed largely by the value of ω. Since points for every 10 degrees of θ are required even for an approximate graph, Δt should be selected as approximately $\frac{\pi}{18\omega}$ seconds. In general the maximum value of the transient current is of most interest, and the spacing of the points may be made accordingly. To illustrate the method, let it be required to find the initial current inrush to a 10-kilovolt-ampere, 2300/230 volt, 60-cycle distribution transformer when its primary is suddenly energized.

Data and Assumed Boundary Conditions:

1. Normal exciting current is 5 per cent of rated full load current.
2. Primary Ri drop is 2 per cent of rated voltage.
3. $N = 1000$ turns.

4. The primary switch is closed at such a time that $\lambda = 0$.
5. The residual magnetism is zero and the flux varies in accordance with the magnetization curve given in Fig. 2 for the first half cycle of the applied emf.
6. Since only the maximum instantaneous value of current is to be calculated hysteresis effects may be neglected.

$$\text{Full-load primary current} = \frac{10{,}000}{2300} = 4.35 \text{ amperes (eff.)}$$

Normal magnetizing current $= 0.05 \times 4.35 = 0.217$ ampere (eff.)

Full load primary RI drop $= 0.02 \times 2300 = 46$ volts (eff.)

$$R = \frac{46}{4.35} = 10.6 \text{ ohms}$$

Assuming an equivalent sine wave variation of magnetizing current, the maximum value of I_m under normal operating conditions is $\sqrt{2} \times 0.217 = 0.307$ ampere. From the magnetization curve, 0.307 ampere establishes a flux of approximately 0.0075 weber or 750,000 maxwells. The maximum value of the Ri drop under steady-state conditions with the secondary on open circuit is:

$$10.6 \times 0.307 = 3.25 \text{ volts}$$

Compared with the maximum value of the applied voltage this represents only one-tenth of one per cent. However, the Ri drop must be included in the transient solution because it is very instrumental in governing the maximum value of the initial current inrush as well as the length of the time required for the circuit to settle into steady-state operation. It may to advantage be neglected during certain parts of the transient period because of its insignificant magnitude.

Neglecting the Ri drop:

$$N\frac{d\phi}{dt} = 3250 \sin \omega t$$

from which:

$$\phi = \frac{3250}{1000}\int \sin \omega t dt + c_1$$

$$\phi = -\frac{3.25}{377} \cos \omega t + c_1$$

Since

$$\phi = 0 \text{ at } t = 0$$

$$c_1 = 0.0086$$

and

$$\phi = 0.0086\ (1 - \cos \omega t) \text{ webers} \tag{13}$$

or

$$\phi = 860{,}000\ (1 - \cos \omega t) \text{ maxwells}$$

If the flux actually varied in accordance with the above expression it would periodically attain maximum values of 0.0172 weber which from the magnetization curve

would require maximum magnetizing currents of approximately 11.6 amperes.[3] Such, of course is not the case because the Ri drop becomes an appreciable factor in limiting the current inrush.

If the Ri drop is neglected for values of current below one ampere a considerable number of steps may be eliminated. At $i = 1$ ampere, the flux established is 0 0125 weber and the length of time required for its establishment may be computed from equation (13).

$$\cos \omega t = -0.453$$
$$(\omega t)_1 = 2.04 \text{ radians or } 117°$$
$$t_1 = 0.00541 \text{ second}$$

From this point on, the step-by-step method will be employed. Δt_2 will be selected as 0.0005 second since it corresponds to a $\Delta\theta_2$ of approximately 10 degrees.

$$\frac{\Delta\phi_2}{\Delta t_2} = \frac{3250}{1000} \sin(\Sigma\omega t) - \frac{10.6 \times i}{1000}$$

The approximations and assumptions made in writing the above expression should be noted in order that successive approximations may, if desired, be made. The finite change in ϕ divided by the finite change in time is set equal to $(1/N)(e - Ri)$ where e, the instantaneous voltage, is considered constant over the intervals and i is considered to be constant at the value which it has at the beginning of the interval. Obviously, these conditions exist only as Δt approaches zero.

The results of the numerical calculations employing the above assumptions are given below in tabular form

Period	$\Sigma\Delta t$ seconds	$\Delta\theta$	$\Sigma\Delta\theta$	$E_m \times$ sin $\Sigma\Delta\theta$ volts	Ri volts	$\Delta\phi$ webers	$\Sigma\Delta\phi$ webers	i amperes
1	0 00541	117°	117°	.. ...	0	0.01250	0 01250	1
2	.00591	10 8°	127 8°	2570	10.6	.00128	.01378	4.07
3	.00641	10 8°	138 6°	2150	43 1	.001055	.01483	6 60
4	00691	10.8°	149 4°	1625	70 0	.000777	.01561	8 46
5	.00741	10 8°	160 2°	1105	89 6	000507	.01612	9 68
6	.00791	10 8°	171 0°	510	102 6	.000204	.01632	10.2
7	00823	7.0°	178 0°	111 8	107 8	negligible	.01632	10 2
8	00873	10 8°	188 8°	−497	107 8	−.000303	.01602	9 44
9	.00923	10 8°	199.6°	−1090	100 0	− 000595	.01543	7 80
10	.00973	10 8°	210 4°	−1635	82 6	− 000860	.01457	5 74

[3] Extrapolating $\phi - i$ curve: The point at which $\phi = 0.0172$ weber is found to be approximately 11.6 amperes. The slope of the curve is assumed constant for the range from 2 to 11.6 amperes. The slope is then:

$$\frac{\Delta\phi}{\Delta i} = \frac{0.0172 - 0.0132}{11.6 - 2} = \frac{0.004}{9.6}$$

It will be more convenient to use the reciprocal of the slope, however.

$$\frac{\Delta i}{\Delta\phi} = \frac{9.6}{0.004} = 2400$$

The initial current inrush in this particular case reaches the maximum instantaneous value of 10.2 amperes, which is approximately 33 times the maximum value of the steady-state exciting current. This maximum is reached at approximately 0.008 second, which in angular measure corresponds to 175 degrees. Having reached its maximum value the current recedes rapidly. Calculation of instantaneous values beyond this maximum is complicated by hysteresis effects in the iron.

Effect of Residual Magnetism.—Residual magnetism, if present, will greatly affect the magnitude of the transient current. The numerical problem given above illustrates the manner in which the Ri and the $N\frac{d\phi}{dt}$ counter-voltages combine to balance the applied emf for the case of zero residual magnetism. A given change in flux is required during the first half cycle after the switch is closed in order to maintain equilibrium. If, instead of being zero, the residual magnetism is +0.005 weber, for example, a very much larger current is required to effect the same change in ϕ. Of course, under these conditions the required change in ϕ is somewhat reduced by virtue of the increase in the Ri drop.

The increase in current due to the presence of the positive residual magnetism may be estimated by the method of finite differences by assuming that the ϕ vs. i curve starts with $\phi = +0.005$ weber at $i = 0$ and gradually approaches the straight-line portion of the magnetization curve. It is instructive to trace out the successive cycles of ϕ vs. i from $t = 0$ until steady-state operating conditions are reached, assuming a small hysteresis effect to be present in the iron. Obviously, in this particular problem, a negative residual magnetism will cause a reduction in the magnitude of the transient current.

Time of Closing the Circuit.—In Chapter VII it was shown that the expression for current in the RL circuit takes the following form when R and L are constant.

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) - \frac{E_m}{Z}\sin(\lambda - \theta)\epsilon^{-\frac{Rt}{L}}$$

Under these conditions the maximum transient component of current occurs when

$$(\lambda - \theta) = \frac{\pi}{2}, \frac{3\pi}{2}, \text{etc.}$$

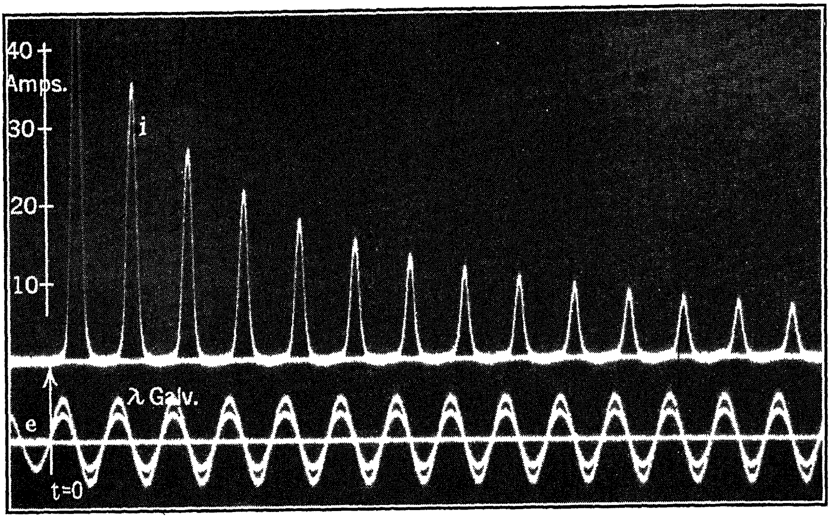
Since θ $\left(\text{equal to } \tan^{-1}\frac{\omega L}{R}\right)$ is very nearly 90 degrees in the ordinary iron-clad circuit, it follows that the maximum transient component occurs when λ is approximately equal to zero or 180 degrees. Although the resultant current in the circuit is not necessarily a maximum because the transient component is a maximum, for practical purposes such is the case.

Likewise it may be shown that minimum transient disturbance occurs in highly inductive circuits when λ is very nearly equal to $\pi/2$, $3\pi/2$, etc.

Oscillographic Illustrations.—Oscillograms 6, 7, and 8 illustrate the general nature of the transient current in iron-clad circuits. They are

Thus for an increase of ϕ of 1 weber beyond 0.0132 weber, a magnetizing current of 2400 amperes is required. It will be necessary to multiply $\Delta\phi$ by 2400 to get the corresponding increase in Δi.

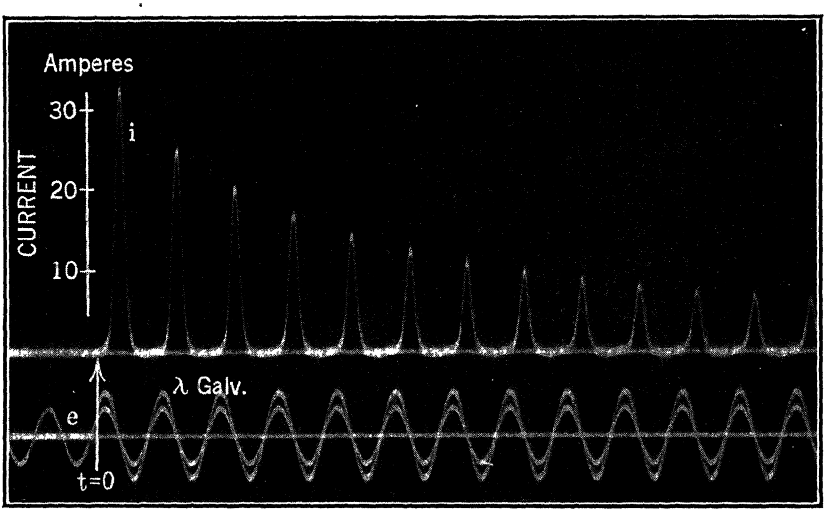
$$\Delta i = 2400\Delta\phi$$



OSCILLOGRAM 6.

Current inrush to an ordinary 2-kilovolt-ampere distribution type transformer

i = current variation.
e = 60-cycle, applied emf. E (eff.) = 234 volts.
λ = recording galvanomenter for determining $t = 0$. (See Fig. 4.)
$\lambda = 0°$, approximately.



OSCILLOGRAM 7.

Transformer current inrush (similar to Oscillogram 6 except for the time of closing the switch).

$\lambda = 30°$, approximately.

fairly representative of average transformer current inrushes. The residual magnetism was in each case reduced to zero prior to the application of the potential difference. The effect of closing the switch at different points along the voltage wave may be observed by direct comparison of the oscillograms since the current calibrations are the same. The oscillographic record of the current variation cannot be used to indicate the point on the voltage wave at which the switch is closed because of the very low value of i near $t = 0$. The time of closing the switch is indicated much more accurately by means of the arrangement shown in Fig. 4. The record of the indicating galvanometer is the larger of the two sinusoidal variations shown directly below the current graphs in Oscillograms 6, 7, and 8.

There is an indication on Oscillogram 6 that the circuit was insecurely closed just prior to $e = 0$ and then became securely closed at $e = 0$.

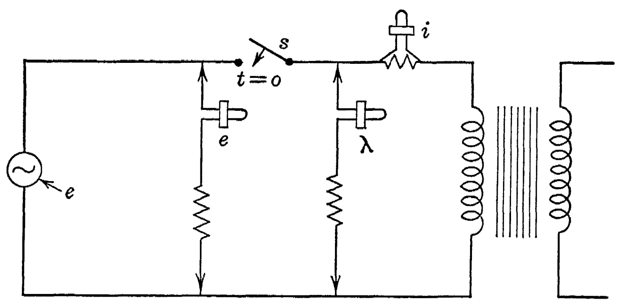
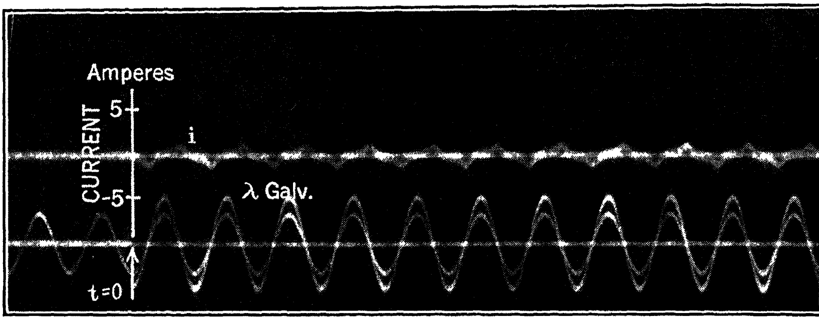


FIG. 4.—Placement of the recording galvanometer for determining the point of $t = 0$ on Oscillograms 6, 7, and 8.

During this period the iron may have become partially magnetized. In any event the first maximum of the current inrush is shown to be more than forty times as large as the maximum steady-state value. The latter is shown to the same scale is Oscillogram 8 where the transient disturbance is minimized by closing the switch at the point of maximum voltage. The minute changes in current caused by the magnetism acquiring its cyclic variation cannot be seen on the oscillogram.

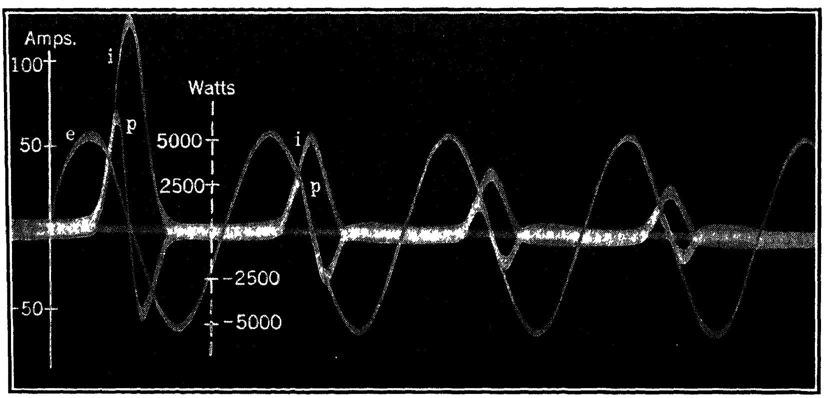
Power Transients.—The instantaneous current and power taken by a transformer primary when suddenly energized is shown in Oscillogram 9. The maximum instantaneous values of current and power would have been somewhat greater than those illustrated if the switch had been closed on the zero point of the emf wave. For the particular case shown the first maximum indicates an instantaneous power of 6250

watts. The average steady-state power (not shown on the oscillogram) is 30 watts. To outward appearances the power graph is a 60-cycle-per-second variation during the early stages of the transient period, which of



OSCILLOGRAM 8.

Initial transformer current when the circuit is closed at the point of approximately maximum voltage. Compared with Oscillograms 6 and 7 a minimum of transient disturbance is indicated in this oscillogram.



OSCILLOGRAM 9

Current and Power delivered to a transformer primary when it is suddenly energized.

e = 60-cycle applied emf. E (eff.) = 114 volts.
i = Current variation. Peak value = 125 amperes.
p = Power variation. Peak value = 6250 watts.
λ is approximately zero.

course is not the case. Alternate cycles of the power variation are very small, so small in fact that their cyclic variation is barely discernible on the oscillogram.

Effect of Variable Inductance and Hysteresis upon Steady-State Phenomena.—Steady-state conditions were not reached on Oscillograms 6, 7, and 9. The current variation illustrated in Oscillogram 8, though practically that of the steady state, cannot be interpreted on account of the small amplitude. Variable inductance and hysteresis also influence the steady-state phenomena. Under steady-state conditions the Ri drop is negligibly small. As shown in the illustrative example, its maximum value was only one-tenth of one per cent of the maximum of the applied emf. Even though the two maximums do not coincide in time phase the Ri drop may be neglected without serious error. Then:

$$N\frac{d\phi}{dt} = E_m \sin(\omega t + \lambda)$$

λ is not a significant factor under steady-state conditions as it merely defines a reference time on the voltage wave for any particular discussion. For convenience here, it is set equal to $\pi/2$.

With $\lambda = \pi/2$:

$$N\frac{d\phi}{dt} = E_m \cos \omega t$$

from which:

$$\phi = \frac{E_m}{\omega N}\sin \omega t$$

$$\phi = \phi_m \sin \omega t, \text{ under steady-state conditions.}$$

Neglecting the Ri drop, the flux must vary sinusoidally in order to maintain equilibrium in the circuit if e varies cosinusoidally. But as the flux is varying sinusoidally with respect to time, the ϕ vs. i variation is that of the well-known hysteresis loop. The corresponding i vs. time graph may be determined by a step-by-step process. Considering the shape of a typical hysteresis loop it is obvious that the i vs. time graph will be something other than a sinusoidal variation. As a first approximation the current vs. time graph takes the following form:

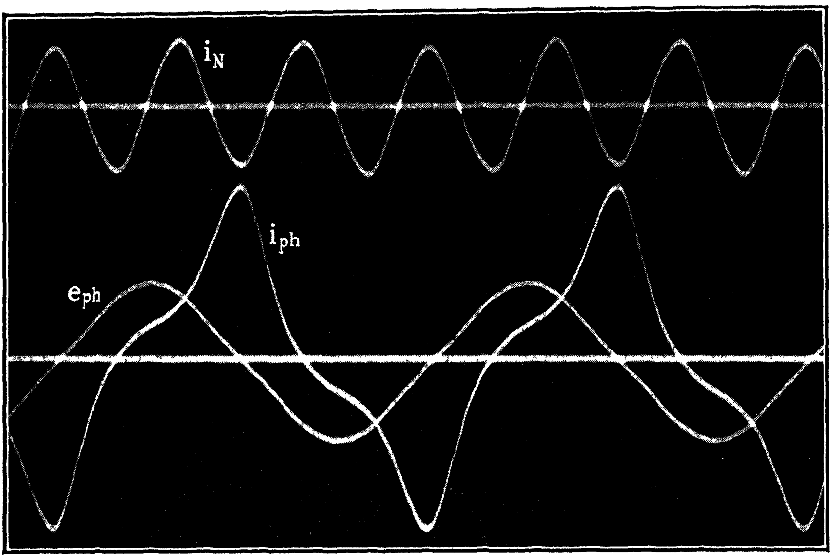
$$i = I_{m1}\sin \omega t + I_{m3}\sin(3\omega t + 90°)$$

Odd harmonics, higher than the third but of lesser magnitude, are present in the actual current variation. The I_{ph} graph of Oscillogram 10 illustrates the irregular cyclic variation of the steady-state current in the primary of a transformer, the secondary of which is open. An

analysis of the wave shows that it is represented to a close degree of accuracy by the following equation:

$$i = I_{m1} \sin(\omega t - 15°) + 0.31 I_{m1} \sin(3\omega t + 92°)$$
$$+ 0.0791 I_{m1} \sin(5\omega t - 145°) + 0.027 I_{m1} \sin(7\omega t - 33°)$$

The relative magnitudes and time phase positions of the various components which combine to form the resultant current graph may be



OSCILLOGRAM 10.

The wave form of the steady-state variation of the exciting current of one transformer of a wye-wye bank.

e_{ph} = 60-cycle, applied emf. E (eff.) = 115 volts.
i_{ph} = steady-state exciting current (eff. value = 0.82 amp.).
i_N = triple harmonic current in the neutral connection (eff. value = 0.8 amp.).

determined by examining the above equation. It will be observed that the effect of the variable inductance and hysteresis has caused a third harmonic to appear in the current, the maximum amplitude of which is nearly one-third as large as the maximum of the fundamental component.

The I_{ph} graph of Oscillogram 10 is the steady-state current of one of the transformers of a wye-wye bank. Since the primary neutral of the bank is connected to the neutral of the wye-connected generator shown in Fig. 5, the operation as regards one transformer is essentially

single phase. The triple harmonic components in the currents of the three transformers combine in time phase in the neutral wire, the oscillographic record of which is shown at the top of Oscillogram 10.

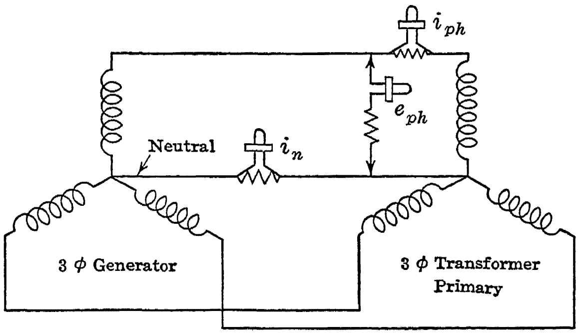


FIG. 5.—Circuit arrangement employed in obtaining the i_{ph}, e_{ph}, and i_n graphs shown in Oscillogram 10.

TRANSIENTS IN ALTERNATING-CURRENT MOTORS

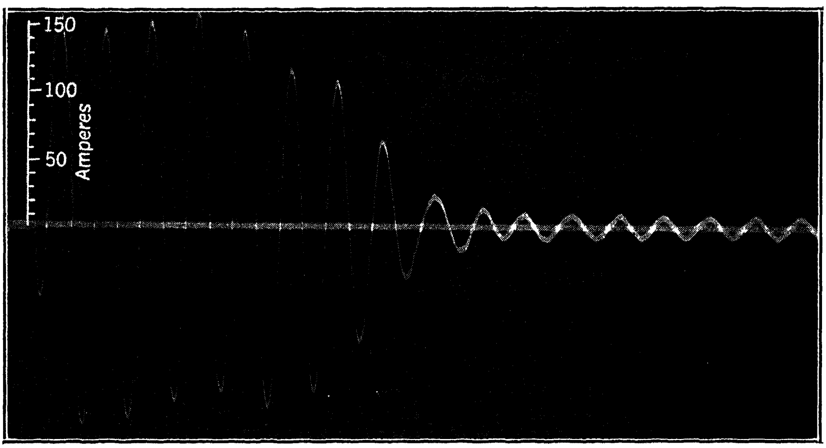
The starting transients in rotating machinery are greatly influenced by the inertia of the rotating member and the connected load. The practical method of investigating the phenomena is by means of the oscillograph. But in the case of large motors the time of the transient period is relatively long and cannot be shown to a satisfactory scale on the printed page. Therefore the oscillographic illustrations of the starting transients that are given here are confined to small machines.

Example 1. No Load Starting Transient Current of a Three-Phase Induction Motor.—Oscillogram 11 shows the transient current in one phase of the primary of a three-phase 5-horsepower, squirrel-cage induction motor. It illustrates the variation in the current during the period in which the motor accelerates from standstill to practically no load operating speed. The maximum instantaneous values of current near $t = 0$ are approximately fifteen times as large as the maximum instantaneous values after operating speed is reached. The transient period is relatively short as compared with the time required for other classes of a-c motors to come up to speed and it is, of course, much shorter than it would be if the motor were connected to a load.

Example 2. Primary and Secondary Current in a Small Single-Phase Induction Motor.—The theory of the operation of a single-phase induction motor even under steady-state conditions is intimately connected with and dependent upon the current in the squirrel-cage secondary. A specially designed stationary cage is required if the secondary

currents are to be recorded oscillographically. Such a machine was employed to obtain simultaneous photographic records of primary and secondary currents of a single-phase induction motor. These records are shown in Oscillogram 12, which is composed of two sections. $t = 0$ is indicated at the left of the upper section.

The motor starts as a split-phase induction type, and during the early part of the transient period both the main primary winding and the auxiliary winding are energized. When a speed, somewhat less than operating speed, is reached a centrifugal device opens the auxiliary



OSCILLOGRAM 11.

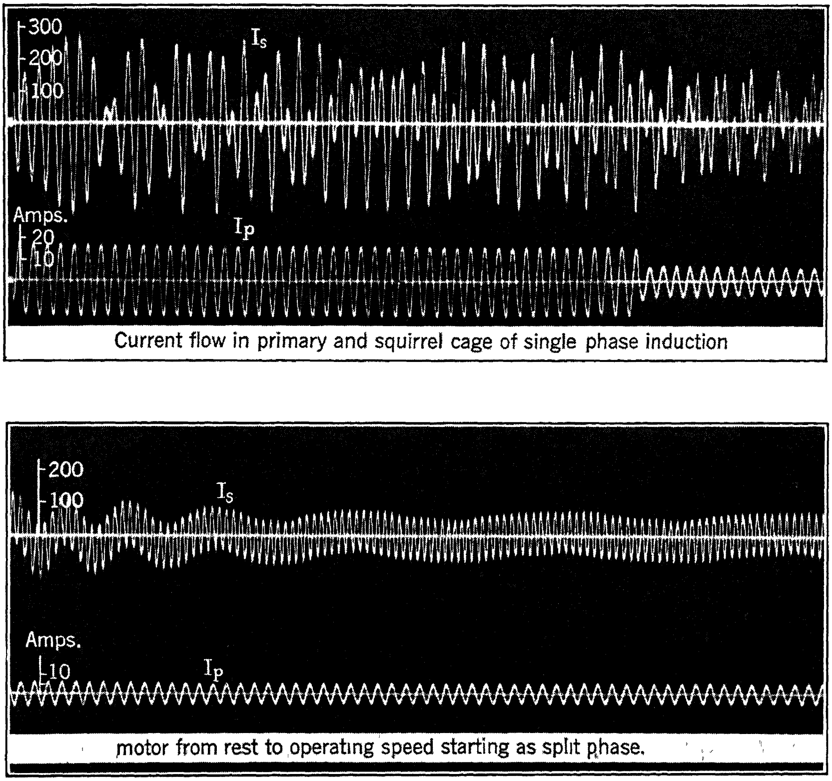
No load starting transient current in a three-phase, 5-horsepower induction motor.

winding circuit. The time at which this occurs is plainly indicated on the I_p graph by the point of sudden decrease in the primary current.

The upper graph of the oscillogram illustrates the transient as well as the steady-state variation in the squirrel-cage secondary current. During the transient period the phenomenon is somewhat complicated by reason of the acceleration of the rotating member. It will be observed that the instantaneous peaks of the secondary current during this period reach magnitudes of approximately 300 amperes. As steady operating speed is reached the squirrel-cage current becomes a double-frequency variation (with respect to the frequency of the primary current) superimposed on a low slip frequency component.

Example 3. Starting Current and Power of a Small Self-Excited Synchronous Motor.—The transient inrush of current to a small synchronous motor is shown in Oscillogram 13. This type of motor comes up to speed as a split-phase induction motor and locks into

synchronism by virtue of the salient pole construction of its secondary member. The time at which the auxiliary primary winding is open-circuited is indicated as well as the relative magnitudes of the starting and steady operating currents. After the auxiliary winding is opened



Current flow in primary and squirrel cage of single phase induction motor from rest to operating speed starting as split phase.

OSCILLOGRAM 12.

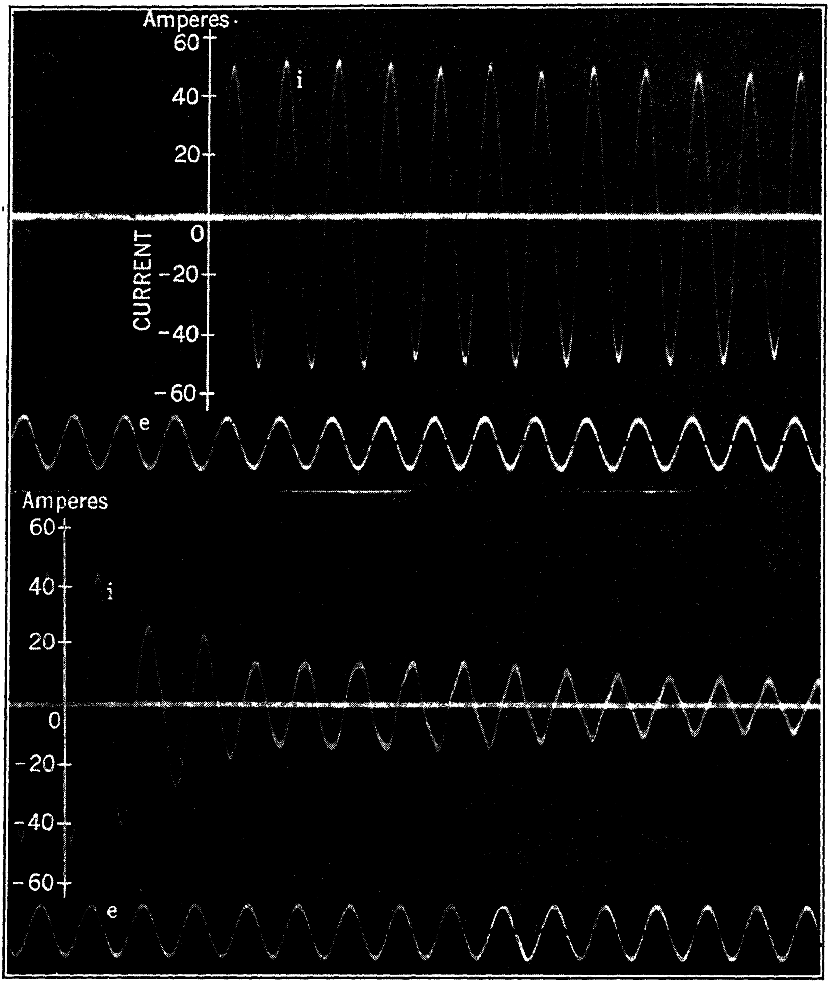
Primary and secondary currents of a specially constructed single-phase induction motor.

I_p = the primary line current.
I_s = the squirrel-cage current.

the wave form of the primary current, after several changes in form, acquires a triangular shape.

Oscillogram 14 shows the power taken by the motor as it comes up to speed. The instantaneous peaks of power during the early transient period represent approximately 4000 watts. A great reduction in the peak values of power as well as in the average power takes place as the

motor approaches steady operating speed. The double frequency of the power variation may be observed throughout the starting period.



OSCILLOGRAM 13.

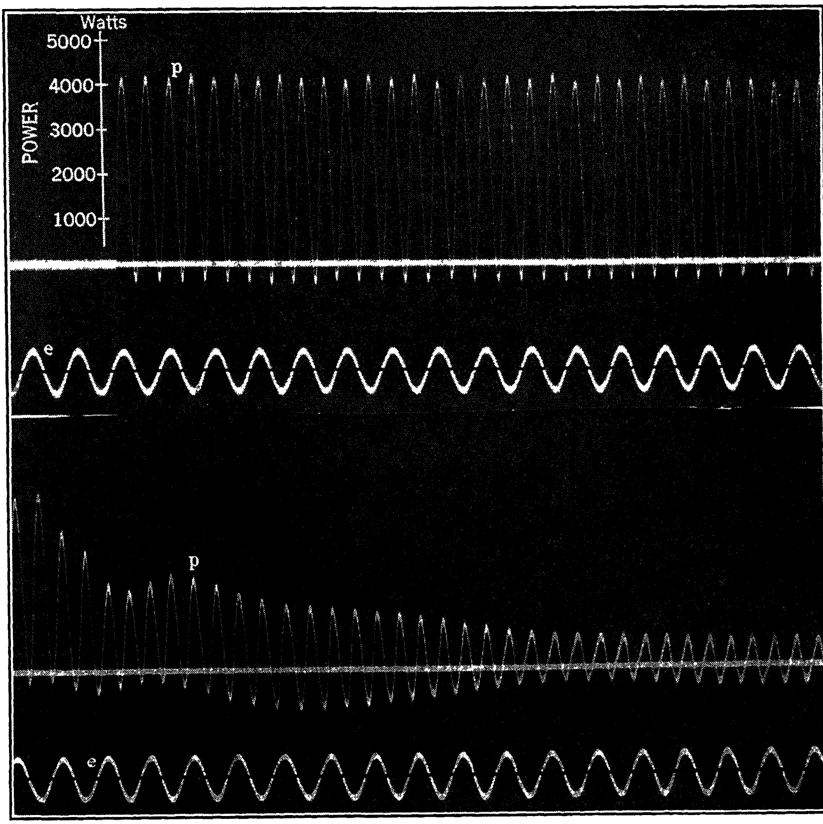
Starting current of a ⅛-horsepower, 115-volt, single-phase synchronous motor. Non-excited field type.

i = primary current.

e = 60-cycle, applied voltage. E (eff.) = 115 volts.

Example 4. The Watt-Galvanometer Method of Studying Motor Transients.—Simultaneous oscillographic records of the voltage, current, and power delivered to an alternating-current motor during a transient

period are very informative. The peak values of current and power may be determined directly, and the various changes that take place may be observed. The change in power factor from cycle to cycle may be determined. The average " true " power over a particular cycle may



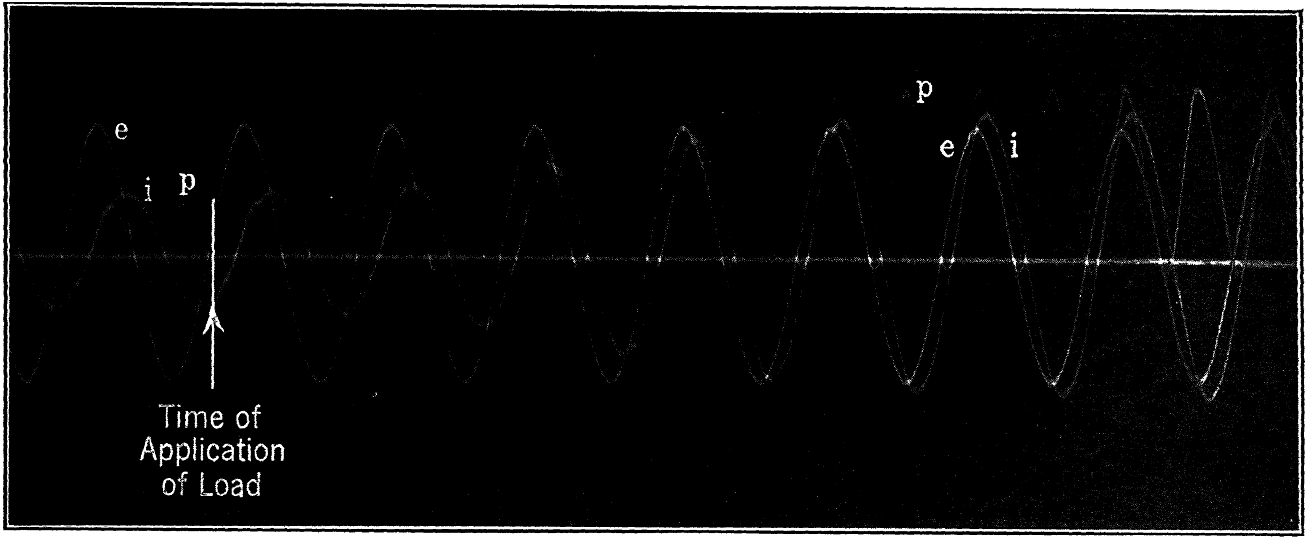
Oscillogram 14.

Power taken by a small synchronous motor (same motor as described in connection with Oscillogram 13).

p = power variation.

e = 60-cycle, applied voltage. E (eff.) = 115 volts.

be determined from the average ordinate of the instantaneous power wave. The average " apparent " power may be determined by multiplying the effective voltage by the effective current. In many cases the effective voltage is simply a constant, and the effective current



OSCILLOGRAM 15.

Illustrating the change in phase current, phase power, and phase voltage of a three-phase, squirrel cage, induction motor as the load changes from no load to rated full load.

e = applied phase voltage. $E_{\text{eff.}}$ = 135 volts.
i = line current. $I_{\text{eff.}}$ = 6 amperes at no load.
I_{eff} = 13 amperes at full load.
p = phase power. $P_{\text{avg.}}$ = 234 watts at no load.
$P_{\text{avg.}}$ = 1580 watts at full load.

over a particular cycle may be determined by graphical means if analytical determination is not feasible.

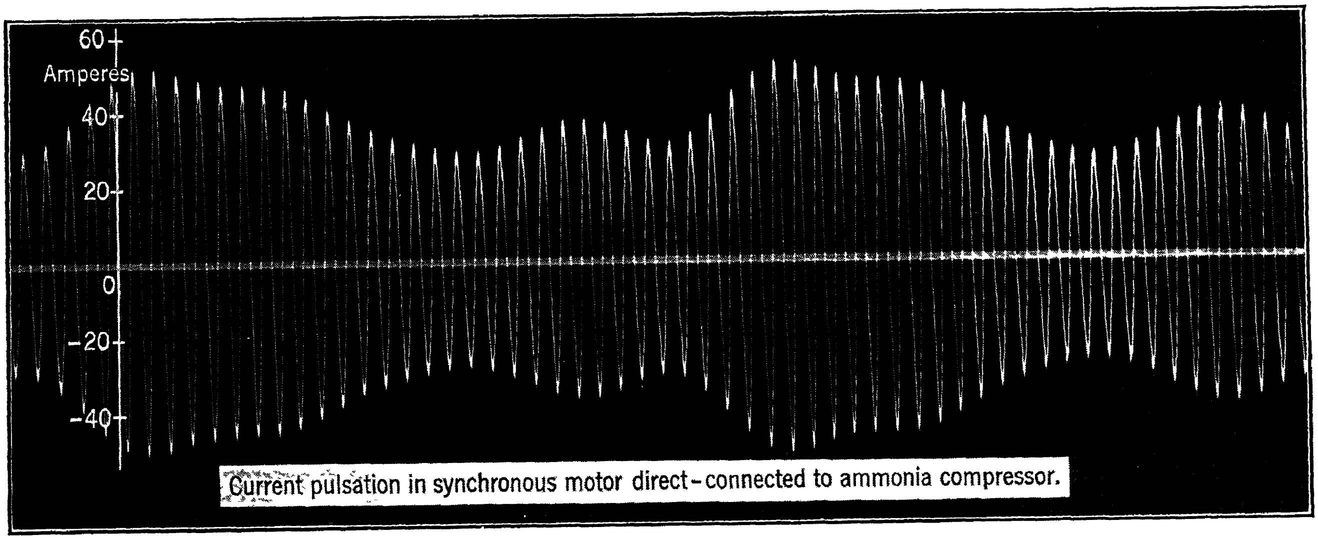
If the transient period is several times as long as the period of one steady-state cycle, and if the transitions in current and power are "smooth," a simple, although approximate, method of analysis may be employed. Oscillogram 15 illustrates the transient variations which accompany the act of applying rated full load to a three-phase induction motor. Single-phase values are shown on the oscillogram. The graph marked e represents the voltage from one line to an artificial neutral or the equivalent-wye phase voltage of the three-phase circuit. The i graph represents the line current which is, of course, the equivalent-wye phase current. The p graph represents $e \times i$ or single-phase power. The average "true" power per phase may be represented by a curve which is midway between the envelopes of the power curve. Assuming sinusoidal variation of e and i, the "apparent" power per phase may be calculated from E_m and I_m.

It is suggested that the student actually construct the current and power envelopes and calculate the power factor at no load, at full load, and at some intermediate point. Vector diagrams of E_{eff} and I_{eff} at the different points are instructive even though the diagram at the intermediate point is not an accurate representation.

Example 5. Transient Effects Caused by Variations in Connected Load.—Fluctuations in the connected load will, of course, cause corresponding fluctuations in the current and power taken by the driving motor. Under certain conditions serious difficulties may arise when the driving motor is of the synchronous type. If fluctuating loads are anticipated the synchronous motor will, in general, be designed with a heavy rotor and an amortisseur winding. Oscillogram 16 illustrates the current pulsations in a synchronous motor which is directly connected to a compressor type of load. The continual change in the magnitude of the supply current may be observed. In this particular case the phenomenon repeats itself every one-half second, approximately, thereby indicating the length of time of the work cycle.

TRANSIENT SHORT-CIRCUIT CURRENTS IN ALTERNATORS

If all the factors affecting the short-circuit currents in a three-phase alternator are considered the subject becomes somewhat involved. Consideration must be given to the relative space positions of the field and armature structures throughout the transient period. However, the problem of alternator short-circuit currents is one of the most commonly discussed space-time transients, and several compre-



OSCILLOGRAM 16.*

Primary line current of a three-phase, 225-horsepower, 184-kilovolt-ampere, 4000-volt, 60-cycle, synchronous motor direct-connected to an ammonia compressor. The compressor operates at 120 r.p.m. and is of the duplex double-acting type. The input current, when the motor is operating at 85 per cent of its rated full-load value, is shown on the oscillogram.

* Taken in conjunction with Professor H. R. Reed, Michigan College of Mining and Technology.

hensive treatises are available for student reference.[4] It is suggested that the student make a detailed study of symmetrical three-phase transient short-circuit currents in three-phase alternators in accordance with the following outline.

1. The assumptions which are made when the initial short-circuit current is expressed as:

$$i = \frac{E_m}{\sqrt{R^2 + X_L{}^2}}\left[\sin(\omega t + \lambda - \theta) - \sin(\lambda - \theta)\epsilon^{\frac{-Rt}{L}}\right]$$

where R is the resistance of the armature winding per phase,

and $X_L = \omega L$, the leakage reactance of the armature winding per phase.

The *initial* short-circuit period may, for the purposes of this study, be considered as the first one or two cycles after $t = 0$.

2. The effect of the mutually coupled field winding (and amortisseur winding if present) upon the initial current predicted by the above equation.

3. The establishment of the rotating magnetic field and the armature reaction effects caused thereby.

4. The effect of the direct-current component in the initial short-circuit current upon the main field flux.

5. The reason for a fundamental frequency variation appearing in the field current during the transient period.

6. The reason for a double-frequency component in the armature short-circuit current of the alternator.

7. The relative magnitudes of the initial short-circuit current and the sustained short-circuit current.

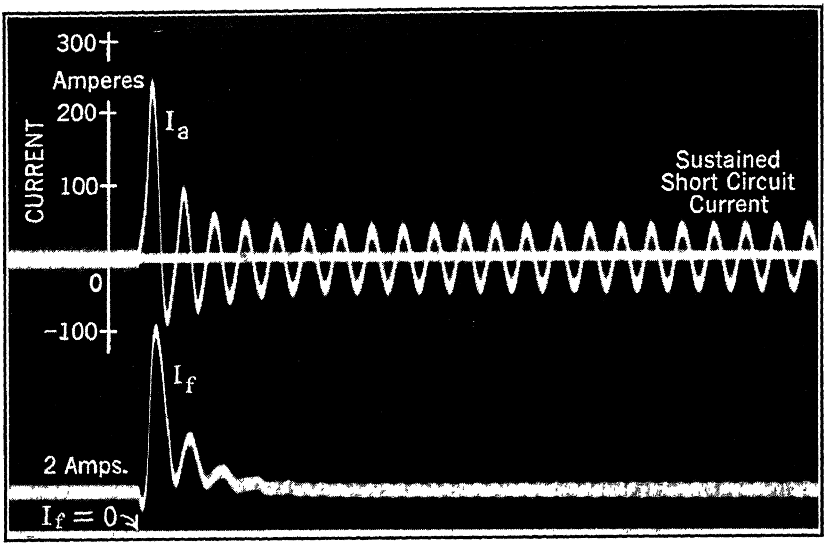
Oscillographic Illustrations.—Three types of short-circuit currents in alternators are shown in Oscillograms 17, 18, and 19. A symmetrical three-phase short is shown in 17, a line-to-line short in 18, and a line-to-neutral short in 19. The alternator employed was a three-phase, wye-connected laboratory machine. Its wye-connected rating is 190 volts, 23 amperes, 7.5 kilovolt-amperes, 60 cycles.

On each oscillogram the variations in the armature current and the field current are shown. The three cases may be compared directly since the galvanometer adjustments are the same for each of the three

[4] Steinmetz, "Transient Electric Phenomena and Oscillations," Chapter 14. Lawrence, "Alternating Current Machinery," Chapter 8. Doherty and Shirley, "Reactance of Synchronous Machines and Its Applications," *Trans. A.I.E.E.*, Vol. 37.

oscillograms. Several characteristic features of alternator short-circuit currents are illustrated.

1. The maximum value of the short-circuit current in the early transient period is about the same regardless of the type of short circuit. It is in this region that the current is governed to a great extent by the leakage reactance of the alternator winding. The point on the voltage wave at which the short circuit occurs will affect the magnitude of the



OSCILLOGRAM 17.

Effect of placing a symmetrical three-phase short circuit on a 7.5-kilovolt-ampere wye-connected alternator. Open circuit voltage = 50 per cent rated voltage.

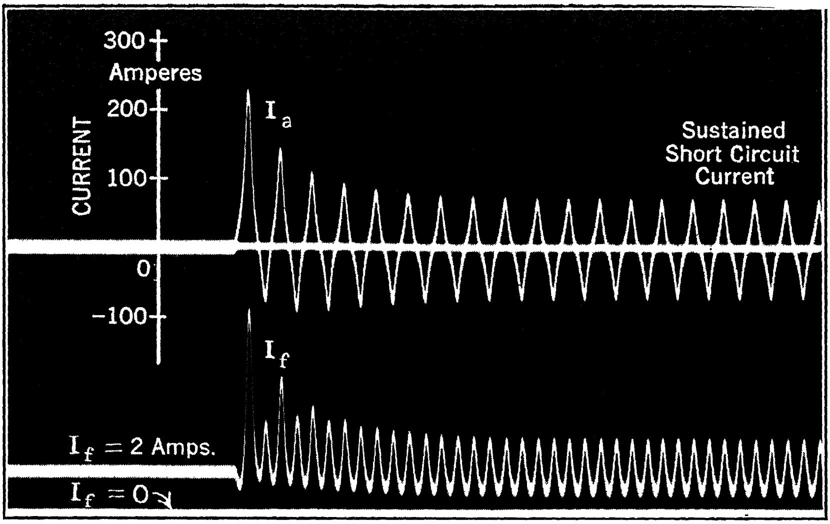
I_a = short-circuit current in one line. Sustained short-circuit current = 31 amp., effective.

I_f = field current, steady-state value = 2 amp.

transient current to some extent, but this point is not indicated on the oscillograms.

2. The ratio of the initial short-circuit current to the sustained short-circuit current differs widely with the different types of short circuits. Peak currents vary from 240 to 43 amperes in the symmetrical three-phase short circuit; from 225 to 70 amperes in the line-to-line short circuit; and from 280 to 115 amperes in the line-to-neutral short circuit. The above changes in current are due largely to the change in the armature reaction with the different types of short circuits.

3. In the symmetrical three-phase short the fundamental fre-

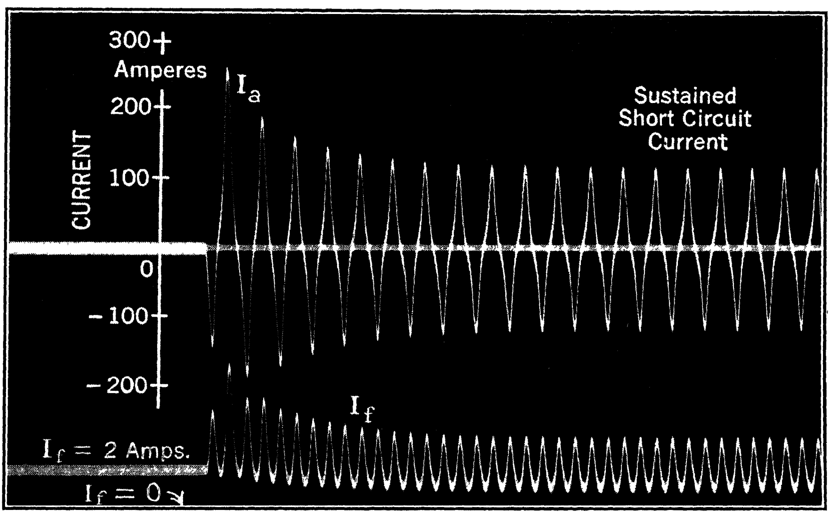


OSCILLOGRAM 18.

Effect of placing a line to line short-circuit on a three-phase, 7.5 kilovolt-ampere. wye-connected alternator. Open-circuit voltage = 50 per cent rated voltage.

I_a = short-circuit current in one line. Sustained short-circuit current = 40 amp., effective.

I_f = field current, steady-state value = 2 amp. The current scales are the same as for the symmetrical three-phase short circuit shown in Oscillogram 17.



OSCILLOGRAM 19.

Effect of placing a line to neutral short circuit on a three-phase, 7.5-kilovolt-ampere, wye-connected alternator. Open circuit voltage = 50 per cent rated voltage.

I_a = short-circuit current line to neutral. Sustained short-circuit current = 65 amp., effective.

I_f = field current, steady-state value = 2 amp. The current scales are the same as for Oscillograms 17 and 18.

quency variation in the field current during the transient period is plainly evident.

4. In the line-to-line and line-to-neutral shorts the double frequency variation in the field current may be observed throughout the duration of the short circuit. The alternate high and low peaks of I_f during the transient period indicate that a fundamental frequency variation is also present in I_f.

SOUND WAVES

Sound waves are usually complex in structure, and they are essentially transient in character. Their electrical transmission is a most vital problem to the communication engineer. The curves shown in Oscillogram 20 are the electrical recordings of sound waves [5] produced by a musical instrument. Oscillogram 20*a* illustrates a relatively pure tone. The sound wave, during the interval depicted on the oscillogram, is represented by an electrical current which is a simple recurring function of time. Its instantaneous intensity may, as a first approximation, be represented by the function:

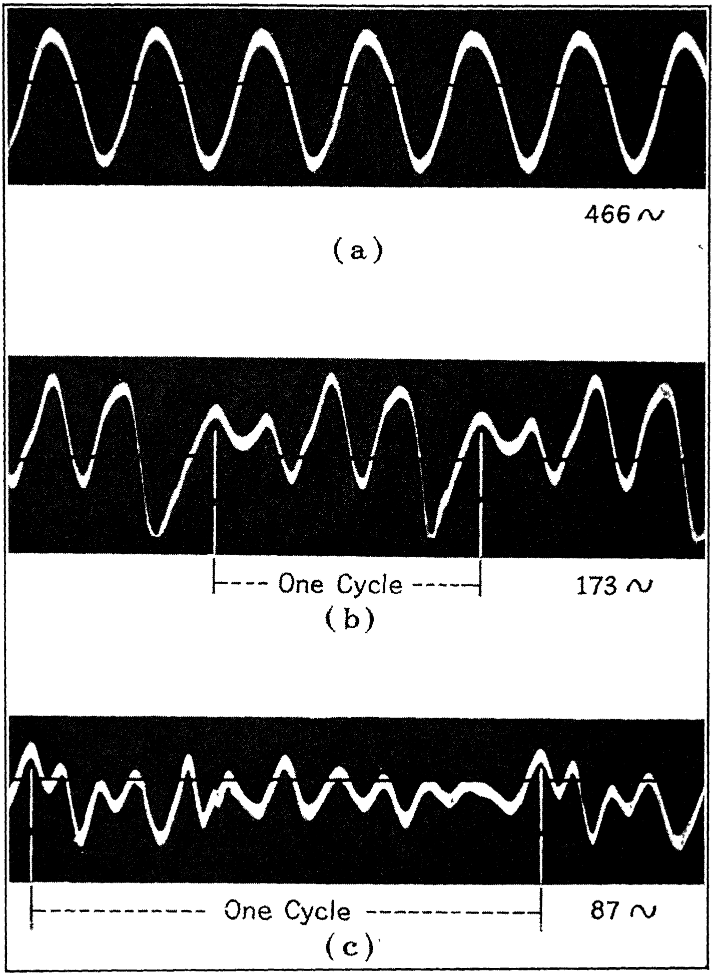
$$K \sin 2926t$$

since the fundamental frequency of the variation is 466 cycles per second. The Fourier series coefficients as determined by a harmonic analyzer are given in the following table.

Component	Frequency, cycles per second	(*A*) Sine component	(*B*) Cosine component	$\sqrt{A^2+B^2}$	Maximum value of component expressed as percentage of maximum value of fundamental
Fundamental . .	466	12 18	0.2	12 18	100 0
2nd Harmonic. .	932	−0 30	−0 31	0 43	3 5
3rd Harmonic.	1398	−0.15	0 33	0.36	3.0
4th Harmonic..	1864	−0 02	0 02	negligible	0.0
5th Harmonic...	2330	0.15	0.02	0 15	1 2

The analysis indicates that the wave is composed chiefly of the

[5] Oscillogram 20, together with the Fourier series analyses, have been furnished by Dr. D. A. Rothschild of the Psychology Department at the State University of Iowa. The analyses were carried out by means of a harmonic analyzer.



Oscillogram 20.

Electrical recordings of three different tones produced by the same wind instrument.

(*a*) represents a relatively pure tone whose fundamental component has a frequency of 466 cycles per second.

(*b*) represents a composite tone whose fundamental frequency is 173 cycles per second.

(*c*) represents a composite tone, the fundamental frequency of which is 87 cycles per second.

fundamental component. This fact is quite obvious from the wave form shown in Oscillogram 20*a*.

Oscillogram 20*b* represents a composite tone which, within the limits of time shown, is very nearly a periodic function of time. Several frequencies are, however, present in the complex wave. An analysis of one complete cycle of the wave shows that its Fourier series coefficients are as follows:

Component	Frequency, cycles per second	(A) Sine component	(B) Cosine component	$\sqrt{A^2+B^2}$	Maximum value of component expressed as percentage of maximum value of fundamental
Fundamental...	173	2 22	−1 55	2 7	100
2nd Harmonic..	346	1 94	−1 92	2 73	101
3rd Harmonic..	519	4 27	2 09	4 75	176
4th Harmonic..	692	5 40	−1 39	5 58	207
5th Harmonic..	865	−0 38	−0 18	0 42	16
6th Harmonic..	1038	0 30	−0 23	0 38	14
7th Harmonic.	1211	0.88	1 02	1 35	50
8th Harmonic..	1384	0 59	−0 07	0 60	22
9th Harmonic...	1557	0 31	0 47	0 56	21
10th Harmonic...	1730	0.05	0 01	0 05	2

It will be observed from the above tabulation that the third and fourth harmonics are the predominating components in the composite tone. A complex sound wave of this nature can be represented by a Fourier series only for short intervals of time because, generally speaking, the function is not continuous. However, an analysis of a typical cycle of a particular tone will reveal the various frequencies that are present and the relative magnitudes of the harmonic components.

Oscillogram 20*c* illustrates a still more complicated wave produced by the same instrument that produced 20*a* and *b*. The fundamental frequency of the *c* graph is 87 cycles per second although it has pronounced second, sixth, and eleventh harmonic components. The fundamental component is relatively small as compared with some of the higher overtones. The eleventh harmonic can be easily singled out by inspection.

Communication systems are designed with a view toward transmitting complex waves, together with the transitions from one complex wave to another, with a minimum amount of distortion. Various

problems arise in this connection since attenuation, in general, depends upon the frequency. If the various components of the complex wave are not transmitted with the same attenuation, the quality of the received wave will differ from that transmitted. Distortion may be so great, under certain conditions, that the received wave becomes an unintelligible signal.

The theory of the propagation of electric waves in wires and cables is founded largely upon the following relations:

$$L \frac{\partial i}{\partial t} dx + Ridx = \frac{\partial e}{\partial x} dx \tag{14}$$

and

$$C \frac{\partial e}{\partial t} dx + Gedx = \frac{\partial i}{\partial x} dx \tag{15}$$

L, R, C and G (leakage conductance) are the *distributed* circuit parameters per unit length of the line and are generally considered to be constant.

x is a measure of distance and is usually considered to be zero at one end of the line. dx thus becomes a differential length of line.

Equation (14) merely states that the differential change in voltage with respect to distance $\left(\frac{\partial e}{\partial x} dx\right)$ is composed of the $\left(L \frac{\partial i}{\partial t} dx\right)$ and the $(Ridx)$ components. It is a statement of Kirchhoff's emf law as applied to a differential element of the line.

Equation (15) describes the manner in which the current in the transmission line changes with respect to distance. The current on one side of a dx element differs from the current on the other side by the sum of the shunt capacitance current $\left(C \frac{\partial e}{\partial t} dx\right)$ and the conductance leakage current $(Gedx)$. The equation is simply a statement of Kirchhoff's current law applied to a differential element of the line.

The general solutions of equations (14) and (15) are not difficult to obtain. By simultaneous solution, (14) and (15) are reducible to two equations, one linear in the dependent variable i and the other linear in the dependent variable e. Particular solutions under various boundary and terminal conditions represent an interesting field of investigation but one which will not be attempted here. The fundamental equations have been stated in order to illustrate how *space* enters into the problem when distributed parameters are involved.

EXERCISES

1. The resistance of a particular variable R circuit is defined as a function of time during the initial period by the expression, $R = (10 + 5000t)$ ohms, where t is in seconds. The above expression is assumed to be true only between the limits of $t = 0$ and $t = 0.02$ second and when the applied voltage is: $(e = 100 \sin 377t)$. It is assumed that e is applied to the R branch at $t = 0$. Make a graph of the current inrush to this particular R circuit during the first cycle of the e variation.

2. The resistance of the circuit described in the above exercise becomes a periodic function of time after $t = 0.1$ second. After $t = 0.1$ second it is assumed that R varies linearly from 200 ohms at $e = 0$ to 240 ohms at $e = E_m = 100$ volts. The R variation from 240 ohms at E_m back to 200 ohms at $e = 0$ is also assumed to be a linear variation with respect to time. Make a graph of one complete cycle of the current variation after $t = 0.1$ second beginning with a zero point on the voltage wave.

3. Analyze the current wave form found in Exercise 2 by the Fourier (or some equivalent) method. Write the equation for i in terms of sine components.

4. Determine the Frolich equation coefficients, m and n, for the magnetization curve shown in Fig. 2. Inasmuch as the resulting equation is to be employed between the limits of $i = 0$ and $i = 1.2$ amperes, the coefficients, m and n, should be determined on that basis.

5. The RL circuit to which the magnetization curve shown in Fig. 2 applies has a resistance of 10.6 ohms (assumed constant) and is composed of 1000 turns. A steady voltage of 12 volts is applied at $t = 0$. Find the time required for the current to build up to 20 per cent of its E/R value, employing the relation stated in equation (10). Repeat for 40, 60, 80, and 90 per cent. Plot the current-time graph and compare with the 12-volt graph shown in Fig. 2. Give reasons for the differences that exist between the current-time graphs found by the Frolich equation method and the finite difference method.

6. One of the various "finite-difference" methods employs the following procedure in determining the length of time required for the current to rise to a particular percentage of its E/R value in an RL circuit.

(*a*) The basic relation is stated as follows:

$$\Delta t = \frac{L\Delta i}{E - R(i)}.$$

(*b*) Δi is fixed at, say 1/10 or 1/15 of the final E/R value of the current, and the increments are known as Δi_1, Δi_2, Δi_3, etc.

(*c*) Δt is calculated for each Δi The value of (i) in the denominator is $\Sigma\Delta i$ and therefore becomes greater with each additional Δi considered.

(*d*) A certain number of Δt's are required for $\Sigma\Delta i$ to equal a particular percentage of E/R. The average of the Δt's multiplied by the number is taken as the length of time required for the current to build up to the particular percentage in question.

Work out the details of the method by applying it to the RL circuit which is described in Exercise 5.

7. Refer to Fig. 6. R is defined by the plate current-plate voltage curve given in Fig. 6(*a*). The "discharge tube" circuit may be considered as an open circuit until $E_c = 160$ volts, at which time it abruptly short-circuits C and reduces E_c to 40 volts in 0.001 second. At the time that E_c is reduced to 40 volts the "discharge

tube" circuit again open-circuits itself. $C = 10$ microfarads and $E = 200$ volts. It is assumed that $Q_0 = 0$.

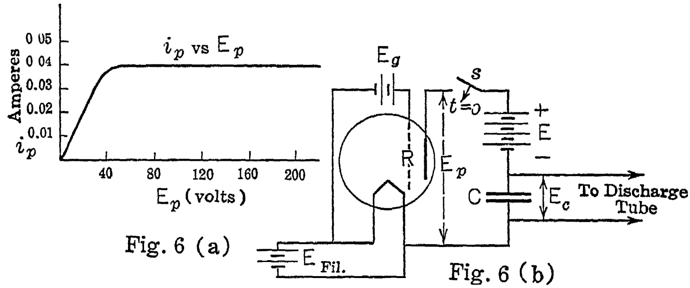


FIG. 6.—(a) represents the plate current-plate voltage curve of the vacuum tube shown in (b). (Screen and suppressor grids not shown.) The circuit arrangement shown in (b) illustrates one method for obtaining a condenser voltage which is practically linear with respect to time.

(a) What is the approximate (constant) value of R between the limits of $E_p = 0$ and $E_p = 40$ volts? Write the expression for R as a function of E_p for values of E_p greater than 40 volts and less than 200 volts.

(b) Draw the graph of E_c versus time for the first three cycles of the E_c variation after the switch of Fig. 6(b) is closed.

(c) Describe one use of this type of circuit.

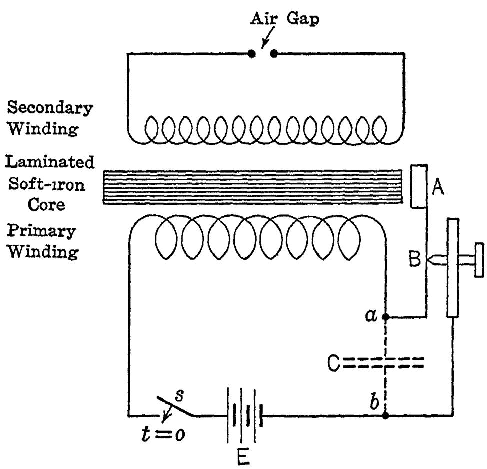


FIG. 7.—The induction coil.

8. (a) Discuss the general nature of the primary and secondary currents in the circuit arrangement shown in Fig. 7 for the period immediately following the closure of the primary switch. Assume that no condenser is present between the points a and b and that the secondary terminals are so adjusted that the rise in magnetic flux does not generate a sufficiently high emf in the secondary to cause a discharge at the secondary terminals. It is assumed, however, that the collapse of the magnetic

flux does induce a sufficiently high secondary voltage to cause a discharge at the air gap. Graph the general trends of the primary and secondary currents with respect to time.

(b) Carry through the above discussion assuming that a condenser is placed between the points a and b.

(c) An approximate comparison of the relative efficiencies of the arrangement shown in Fig. 7 with and without a condenser may be obtained by neglecting the secondary resistance and assuming that the self-inductances and the mutual inductance are constant. With these assumed conditions show that, without a condenser:

$$i_2 \text{ approaches } \frac{M}{L_2} I_0 \text{ as a maximum}$$

where: M is the mutual inductance between windings.

L_2 is the self-inductance of the secondary.

I_0 is the current in the primary at the moment the primary circuit is broken.

Show that, with a condenser:

$$i_2 \text{ approaches } \frac{2M}{L_2} I_0 \text{ as a maximum}$$

9. Discuss the general nature of the current variation in the ALC loop of Fig. 8 during the period immediately following the closure of the switch. It should be

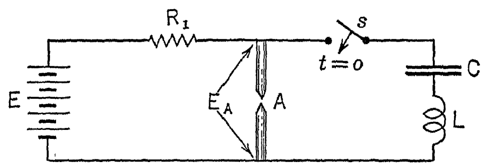


FIG. 8.—The Duddell singing arc.

recognized that the carbon terminals and the arc (A) have relatively large negative temperature coefficients. Under what conditions will a sustained oscillation exist in the ALC loop? What will be the approximate frequency of the sustained oscillation?

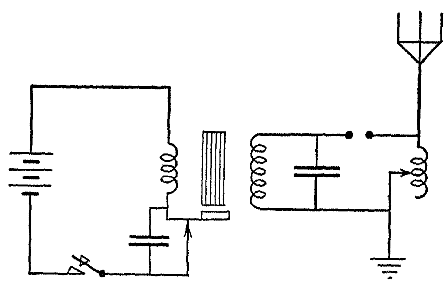


FIG. 9.—Elementary type of spark transmitter.

10. Describe the operation of the spark transmitter shown in Fig. 9. Discuss the nature of the current-time graph of the current in the loop formed by the secondary condenser, the spark gap, and the antenna coil.

11. A voltage, $e = E\epsilon^{-mt}$, is impressed upon a coil having a self-inductance of L henrys and a resistance of R ohms. Derive the equation for current in the circuit assuming that R and L are constant. The factor m governs the rate at which the impulse voltage subsides and is considered to be constant.

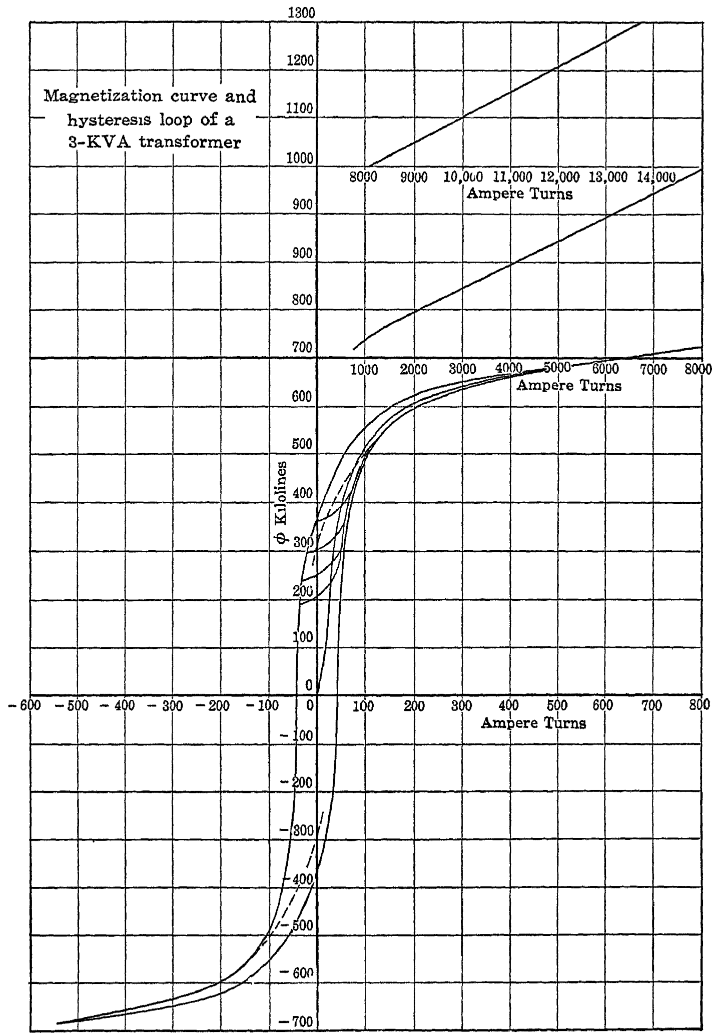


FIG. 10.

12. A voltage, $e = E\epsilon^{-mt}$, is impressed upon an RC series circuit. Derive the equation for current in the circuit assuming that R and C are constant and that the initial condenser counter-voltage, Q_0/C, is less than E in magnitude.

13. An emf, $e = 1000\epsilon^{-5t}$, is impressed upon an RLC series circuit. $R = 240$

ohms, $L = 100$ millihenrys, and $C = 10$ microfarads. What is the voltage across the condenser and the current in the circuit at 0 001 second after closing the switch? It is assumed that the circuit is at rest at $t = 0$ and that $Q_0 = 0$.

14. The magnetization curve and hysteresis loop shown in Fig. 10 apply to a particular 3-kilovolt-ampere, 220-volt, 60-cycle, single-phase transformer. The transformer has 156 primary turns, and the resistance of the primary winding, including the line wires, is 0.185 ohm. Plot two cycles of the starting current curve when the impressed voltage is $e = \sqrt{2} \cdot 220 \sin 377t$, and the residual flux is +360,000 maxwells. It is assumed that the secondary of the transformer is on open circuit.

15. A 17-kilovolt-ampere, three-phase, 220-volt, 60-cycle, wye-connected synchronous motor is connected to a three-phase system. The motor resistance is 0.175 ohm, and the phase reactance is 0.855 ohm. Plot the resultant mmf-space curve in rectangular and in polar form for at least three cycles of emf. The steady-state mmf may be taken as 100 per cent. Plot the curve showing the velocity of the resultant mmf as a function of the velocity of the synchronously rotating mmf. (See Steinmetz: "Transient Electric Phenomena and Oscillations," pp. 197–204, third edition)

16. Solve equations (14) and (15) for explicit expressions of i and e in terms of the two independent variables x and t. (See Steinmetz: "Transient Electric Phenomena and Oscillations," pp. 449–458, third edition, or Bewley: "Traveling Waves on Transmission Systems," pp. 7–14.)

APPENDIX

SECTION I

THE DIFFERENTIAL EQUATIONS OF ELEMENTARY CIRCUIT THEORY

Distinguishing Characteristics.—The differential equations of elementary circuit theory [1] are distinguished by five principal characteristics:

1. *They are linear.*—For example, the differential equation,

$$A_1 \frac{d^2y}{dx^2} + A_2 \frac{dy}{dx} + A_3 y = f(x) \tag{1}$$

is linear in the three variables $\frac{d^2y}{dx^2}, \frac{dy}{dx}$, and y, if the coefficients, A_1, A_2, and A_3 do not depend on these variables. An expression similar to the left-hand member of equation (1) is said to be linear in a set of variables if each of the additive terms contains just one of these variables raised to the first power. Circuits that are inherently linear can, of course, be overloaded to a point where they become non-linear. Certain systems, notably those including vacuum-tube devices, are essentially non-linear in character. Such systems require special treatment.

2. *The coefficients are constant.*—This statement is true only within certain limits. The general theory throughout the text, however, is outlined on the basis of constant coefficients inasmuch as the assumption of constant coefficients greatly simplifies the mathematical solutions. Examples of variable coefficients are given in Chapter X.

3. *Several dependent variables are, in general, present.*—For example, the current in each of the various branches of a network may be considered as a distinct "dependent" variable. In the study of elementary transients these currents are dependent upon the "independent" variable, time, for their values.

4. *The simplicity of the boundary conditions.*—The boundary conditions are in general simple of statement; either the system is "at rest"

[1] T. C Fry, "Elementary Differential Equations." Conkwright, "Differential Equations." Franklin, "Differential Equations for Electrical Engineers." T. C. Fry, "Differential Equations as a Foundation for Electrical Circuit Theory," *American Mathematical Monthly*, Vol. 36.

at a certain instant, or else it is in a "steady state." Examples of boundary conditions other than those stated above may be found in Chapter III. They are more of academic interest than of practical value.

5. *The nature of the applied voltage (the driving force of the circuit) is of simple mathematical form.*—It is either of the simple harmonic (a-c) type, $E_m \sin(\omega t + \lambda)$, or of the constant (d-c) type. Momentary impulses of voltage in the form of natural or artificial lightning are sometimes encountered in particular cases. In other instances, for example, in the determination of subsidence currents, the applied voltage is zero.

Definitions

A differential equation is an equation involving derivatives or differentials with or without the variables from which these derivatives or differentials are derived.

An ordinary differential equation is one in which the derivatives involved are taken with respect to a single independent variable. The equations encountered in elementary circuit analysis are of the ordinary type since only one independent variable, namely, time, is considered.

A partial differential equation is one which contains partial derivatives. The fact that partials are present indicates the existence of two or more independent variables to which these derivatives have been formed. Partial differential equations enter into circuit analysis when both time and space are treated as independent variables. Equations (14) and (15) in Chapter X contain partial derivatives.

The order of a differential equation is the order of the highest derivative in the equation. For example, equation (1) is an equation of the second order.

The degree of a differential equation is the degree of the derivative of highest order in the equation after the equation is freed from radicals and fractions in its derivatives. For example,

$$\left(\frac{d^3y}{dx^3}\right)^2 - x^3\frac{dy}{dx} = y \tag{2}$$

is an equation of second degree. Such equations are not encountered in elementary circuit theory. Equation (1) is a first degree equation.

A solution of a differential equation is a relation between the variables of the equation not involving derivatives, such that if the value of the dependent variable is substituted in the equation, the equation is satisfied. In general, the solution of an ordinary differential equation involves finding an explicit expression for the dependent variable in

terms of the independent variable which will satisfy the original equation. Certain types of equations lend themselves to one method of attack whereas other types will require radically different treatment.

A general solution is one which contains arbitrary constants equal in number to the order of the equation.

A particular solution is one obtained from the general solution by giving one or more of the constants particular values. It is somewhat less confusing if this type of solution is called " the solution in a particular case." The term " particular solution " is sometimes used to designate the portion of the general solution which is usually designated as " the particular integral." The " particular integral " is defined in a later paragraph.

The General Solution of a Linear Differential Equation of the First Order with Constant Coefficients

The common form encountered in electric circuit work is:

$$\frac{di}{dt} + ai = he \tag{3}$$

where:

- i is the dependent variable.
- t is the independent variable.
- a and h are constant.
- e defines the nature of the applied voltage. For example, $e = \sin(\omega t + \lambda)$ in the case of alternating voltages. In the case of direct voltages e is merely equal to unity. The magnitude of the applied voltage is contained in the constant h. The more general case in which e varies sinusoidally with respect to t will be considered in the general derivations that follow. In this case e is a function of time.

The general solution of equation (3) is:

$$i = h\epsilon^{-at} \int e\, \epsilon^{at}\, dt + c_1 \epsilon^{-at} \tag{4}$$

The proof of the solution rests in the solution's ability to satisfy the original equation. From the relation stated in equation (4):

$$\frac{di}{dt} = h\epsilon^{-at} \cdot e\epsilon^{at} - ah\epsilon^{-at} \int e\epsilon^{at} dt - ac_1 \epsilon^{-at} \tag{5}$$

$$ai = ah\epsilon^{-at} \int e\epsilon^{at} dt + ac_1 \epsilon^{-at} \tag{6}$$

Adding the right-hand members of (5) and (6) yields (he) which proves that (4) is a general solution of equation (3). The arbitrary constant in the general solution is c_1.

Equation (3) is one of the forms of differential equations which can be reduced to a simple explicit derivative form by multiplying both sides by an "integrating factor." The integrating factor in this case is ϵ^{at}. Thus:

$$\epsilon^{at}\frac{di}{dt} + a\epsilon^{at}i = h\epsilon^{at}e \tag{7}$$

The reason for multiplying through by ϵ^{at} becomes apparent when it is recognized that the left-hand side of the equation can now be reduced to a single derivative.

Let

$$u = i\epsilon^{at} \tag{8}$$

$$\frac{du}{dt} = \epsilon^{at}\frac{di}{dt} + a\epsilon^{at}i \tag{9}$$

Therefore:

$$\frac{du}{dt} = h\epsilon^{at}e \tag{10}$$

$$u = h\int \epsilon^{at}e\,dt + c_1 \tag{11}$$

$$i = u\epsilon^{-at}$$

or

$$i = h\epsilon^{-at}\int \epsilon^{at}e\,dt + c_1\epsilon^{-at} \tag{12}$$

The solution in a particular case will involve the integration, $\int \epsilon^{at}e\,dt$, and the evaluation of the constant of integration, c_1.

(*a*) If $e = \sin(\omega t + \lambda)$ and $h = E_m/L$:

$$\int \epsilon^{at}\sin(\omega t + \lambda)dt = \frac{\epsilon^{at}[a\sin(\omega t + \lambda) - \omega\cos(\omega t + \lambda)]}{a^2 + \omega^2}$$

$$= \frac{\epsilon^{at}[\cos\theta\sin(\omega t + \lambda) - \sin\theta\cos(\omega t + \lambda)]}{\sqrt{a^2 + \omega^2}} \tag{13}$$

where $\theta = \tan^{-1}\omega/a$, and the solution takes the following form:

$$i = \frac{E_m}{L\sqrt{a^2 + \omega^2}}\sin(\omega t + \lambda - \theta) + c_1\epsilon^{-at} \tag{14}$$

(b) If $e = 1$ and $h = E/L$:

$$\int \epsilon^{at}\, e\,dt = \frac{1}{a}\epsilon^{at}$$

and the solution takes the form given below:

$$i = \frac{E}{La} + c_1 \epsilon^{-at} \tag{15}$$

(c) If $e = 0$:

$$\int \epsilon^{at}\, e\,dt = 0$$

and the solution becomes:

$$i = c_1 \epsilon^{-at} \tag{16}$$

The boundary conditions in a particular case will determine the value of the arbitrary constant, c_1.

The basic voltage equations of the RL and the RC series circuits can be reduced to the form shown in equation (3). Therefore the current solutions of these series circuits are contained in the general solution shown in equation (4). The series circuit containing resistance, self-inductance, and capacitance gives rise to a second order differential equation. By proper manipulation, however, the solution may be separated into two parts, each of which is similar to the one shown above.

The General Solution of a Linear Differential Equation of the Second Order with Constant Coefficients

The common form encountered is:

$$\frac{d^2i}{dt^2} + n\frac{di}{dt} + mi = he \tag{17}$$

where n, m, and h are constants, and e defines the mode of variation of the applied voltage which, in the general case, is a function of t.

Let:

$$\frac{d}{dt} = D$$

Equation (17) may then be written in the form:

$$(D^2 + nD + m)i = he \tag{18}$$

If the expression within the parenthesis is factored as an algebraic expression in D the equation takes the form:

$$(D - \alpha_1)(D - \alpha_2)i = he \tag{19}$$

where:

$$\left.\begin{aligned} \alpha_1 &= \frac{-n + \sqrt{n^2 - 4m}}{2} = -a + b \\ \alpha_2 &= \frac{-n - \sqrt{n^2 - 4m}}{2} = -a - b \end{aligned}\right\} \tag{20}$$

The factors $(D - \alpha_1)$ and $(D - \alpha_2)$ are symbolic operators, and the correct interpretation of them must be made if the left-hand member of equation (19) is to remain equivalent to the left-hand member of equation (17). The interpretation of the symbolic operators is given below.

and

$$\left.\begin{aligned} (D - \alpha_1)i &= \frac{di}{dt} - \alpha_1 i \\ (D - \alpha_2)i &= \frac{di}{dt} - \alpha_2 i \end{aligned}\right\} \tag{21}$$

Therefore:

$$(D - \alpha_1)\left(\frac{di}{dt} - \alpha_2 i\right) = he \tag{22}$$

and

$$(D - \alpha_2)\left(\frac{di}{dt} - \alpha_1 i\right) = he \tag{23}$$

The order of operation is immaterial since performing the operations indicated in either (22) or (23) yields:

$$\frac{d^2i}{dt^2} - (\alpha_1 + \alpha_2)\frac{di}{dt} + \alpha_1\alpha_2 i = he \tag{24}$$

α_1 and α_2 are defined by the relations stated in (20).

and

$$\left.\begin{aligned} -(\alpha_1 + \alpha_2) &= n \\ \alpha_1\alpha_2 &= m \end{aligned}\right\} \tag{25}$$

It follows that the left-hand member of (24) is equal to the left-hand member of equation (17). The operations which have been performed and the interpretations given to the symbolic operators are therefore valid.

Solving equation (19) for i, the equation takes the following form:

$$i = \frac{1}{(D - \alpha_1)(D - \alpha_2)} he \tag{26}$$

where the meaning of the symbolic form on the right will be explained presently. Provided that $\alpha_1 \neq \alpha_2$, $\dfrac{1}{(D - \alpha_1)(D - \alpha_2)}$ may be broken up into partial fractions in the same manner as if it were an algebraic expression in D. The separation may be effected as follows:

$$\frac{1}{(D - \alpha_1)(D - \alpha_2)} = \frac{A}{(D - \alpha_1)} + \frac{B}{(D - \alpha_2)} \tag{27}$$

or

$$1 = A(D - \alpha_2) + B(D - \alpha_1) \tag{28}$$

The above relation must hold for all values of D, hence when:

When:
$$\left.\begin{aligned} D &= \alpha_1, \quad A = \frac{1}{\alpha_1 - \alpha_2} \\ D &= \alpha_2, \quad B = -\frac{1}{\alpha_1 - \alpha_2} \end{aligned}\right\} \tag{29}$$

It follows that:

$$\frac{1}{(D - \alpha_1)(D - \alpha_2)} = \frac{1}{\alpha_1 - \alpha_2}\left[\frac{1}{(D - \alpha_1)} - \frac{1}{(D - \alpha_2)}\right] \tag{30}$$

and that:

$$i = \frac{1}{2b}\left[\frac{1}{(D - \alpha_1)} - \frac{1}{(D - \alpha_2)}\right] he \tag{31}$$

$$i = \frac{1}{2b} \cdot \frac{he}{(D - \alpha_1)} - \frac{1}{2b} \cdot \frac{he}{(D - \alpha_2)} \tag{32}$$

It is now plain that the solution of i may now be broken up into two parts such that:

$$i = i_1 + i_2 \tag{33}$$

where:

and
$$\left.\begin{aligned} i_1 &= \frac{1}{2b} \cdot \frac{he}{(D - \alpha_1)} \\ i_2 &= -\frac{1}{2b} \cdot \frac{he}{(D - \alpha_2)} \end{aligned}\right\} \tag{34}$$

$$(D - \alpha_1)i_1 = \frac{di_1}{dt} - \alpha_1 i_1 = \frac{he}{2b} \tag{35}$$

$$(D - \alpha_2)i_2 = \frac{di_2}{dt} - \alpha_2 i_2 = -\frac{he}{2b} \tag{36}$$

Equations (35) and (36) are first order equations, the general solution of which has been given in detail. Employing the general solution of first order equations as given in equation (4) it follows that:

$$i_1 = \frac{h}{2b}\epsilon^{\alpha_1 t}\int \epsilon^{-\alpha_1 t} e\,dt + c_1\epsilon^{\alpha_1 t} \tag{37}$$

and

$$i_2 = -\frac{h}{2b}\epsilon^{\alpha_2 t}\int \epsilon^{-\alpha_2 t} e\,dt + c_2\epsilon^{\alpha_2 t} \tag{38}$$

from which:

$$i = \frac{h}{2b}\left[\epsilon^{\alpha_1 t}\int \epsilon^{-\alpha_1 t} e\,dt - \epsilon^{\alpha_2 t}\int \epsilon^{-\alpha_2 t} e\,dt\right] + c_1\epsilon^{\alpha_1 t} + c_2\epsilon^{\alpha_2 t} \tag{39}$$

Although equation (39) represents the general solution of a second order differential equation, the form, as such, is seldom employed in effecting solutions in particular cases. It is of theoretical interest and importance, however, since it points the way to the general solution of higher order linear differential equations.

Particular Integral and Complementary Function

An examination of the general solution given in equation (39) reveals the fact that one part of the solution involves no arbitrary constants. This part is known as the *particular integral* and represents the " steady-state " solution of i. Such is to be expected since the *particular integral* is independent of the boundary conditions. The *particular integral* is sometimes called the *particular solution.*

The part of the solution of a linear differential equation involving the constants of integration $(c_1\epsilon^{\alpha_1 t} + c_2\epsilon^{\alpha_2 t})$ in equation (39) is called the *complementary function.* It will be observed that the *complementary function* may be evaluated without regard to the nature of the driving voltage, e, except in so far as a consideration of the driving voltage is necessary in the evaluation of the constants of integration. The *complementary function* represents the transient behavior or response of the electric circuit.

General Solution of Higher Order Linear Differential Equations with Constant Coefficients

The method employed in determining the general solution of second order equations is, theoretically, capable of extension. Consider the case of a third order equation:

$$\frac{d^3i}{dt^3} + m_1 \frac{d^2i}{dt^2} + m_2 \frac{di}{dt} + m_3 i = he \tag{40}$$

Following the procedure previously outlined we let:

$$D = \frac{d}{dt}$$

Then:

$$(D^3 + m_1 D^2 + m_2 D + m_3)i = he \tag{41}$$

The roots of $[D^3 + m_1 D^2 + m_2 D + m_3 = 0]$ are evaluated as if it were an algebraic equation. Indicating the roots by α_1, α_2, and α_3 equation (41) may be written as follows:

$$i = \frac{1}{(D - \alpha_1)(D - \alpha_2)(D - \alpha_3)} he \tag{42}$$

Upon separation into partial fractions, three distinct parts of the solution appear. Each of these may be evaluated in accordance with the method outlined for first order differential equations. The complete solution may then be given as the sum of the three component parts.

The most useful form of the general solution is:

$$i = (\text{steady state solution}) + c_1 \epsilon^{\alpha_1 t} + c_2 \epsilon^{\alpha_2 t} + c_3 \epsilon^{\alpha_3 t} \tag{43}$$

The " steady-state solution " may, in general, be determined by simple algebraic manipulations that are well known to the electrical engineer. Examples of the numerical solution of third and fourth order differential equations may be found in Chapters IV and V of the text.

Solution of Ordinary Differential Equations with Variable Coefficients

Consider the basic voltage equation of the simple series circuit, namely:

$$\frac{di}{dt} + (a)i = (h)e \tag{44}$$

In the case of the RL circuit, $(a) = R/L$ and $(h) = E/L$. In the case of the RC circuit, $(a) = 1/RC$ and $(h) = E/R$. If the circuit par-

ameters, namely, R, L, and C, are not constant it is plain that (a) and (h) will not be constant. If the variations in R, L, and C can be written as particular functions of i or t, solutions are sometimes obtainable by ordinary methods. [For example, see Chapter X, equations (1) to (10).] Generally speaking, the solution of differential equations involving variable coefficients is a difficult task. The first problem is to express the variation of the parameters as mathematical functions of the i or t. This in itself often takes the investigation outside the realm of practical mathematics. In most instances, the solutions, if they can be obtained, will not be general solutions since the variation of the circuit parameters will be defined in terms of particular constants rather than arbitrary constants.

When the more refined mathematical methods become too involved a step-by-step method of solution is sometimes employed. One such procedure is described in detail in Chapter X under the heading "The Method of Finite Differences."

SECTION II

HEAVISIDE'S OPERATIONAL CALCULUS

During the course of his various studies Heaviside [1] devised a rather unique form of analysis which has come to be known as Heaviside's operational calculus. Within the past decade, this form of mathematics has gained considerable prominence in the field of electrical engineering. Several interpretations of Heaviside's operational methods have appeared in book and pamphlet form, although at first the reader will be impressed with the apparent divergence of the different interpretations. The fact is that Heaviside concerned himself chiefly with applications and results, paying little attention to the rigorous proofs of the mathematics that he employed. Inasmuch as his results were consistently correct the type of mathematics that he employed became a subject of great interest.

Several rigorous proofs of Heaviside's methods have been worked out. Bromwich,[2] who was one of the first to give a rigorous and systematic proof of these unique methods, employs line integration in a complex plane in his solution. Carson [3] accomplishes the same results by means of the superposition theorem and integral equations. Rigorous treatments of this type require an extended mathematical background and a general proficiency in the subject of mathematics on the part of the reader.

It is the purpose of this brief treatment to explain the operational methods which have been used in the text. No attempt is made to cover the complete subject of operational calculus inasmuch as it has many ramifications. The explanations and proofs contained herein are presented in terms of simple undergraduate mathematics so far as it is reasonably possible to do so. It is hoped that this elementary type of approach will serve as an introduction to the subject and at the same time provide an incentive for the student to make a more thorough and comprehensive study of operational methods.[4]

[1] An interesting personal sketch of Oliver Heaviside may be found in the fore part of Berg's "Heaviside's Operational Calculus."

[2] T. J. I'A. Bromwich, *Proceedings of the London Mathematical Society*, Series 2, Vol. 15, 1916.

[3] J. R. Carson, "Electric Circuit Theory and the Operational Calculus."

[4] Bush, "Operational Circuit Analysis."

The "Heaviside" Condition.—The "unextended" Heaviside method of investigating the behavior of an electric circuit is to apply suddenly unit potential difference to one branch of the circuit and to note the corresponding response in the various parts of the circuit. It is assumed that the circuit is at rest, that is, no current is flowing and no electric charges exist at the time of the application of the potential difference. One method of writing such a procedure into equation form is to set the right-hand side of the equilibrium equation equal to Heaviside's "unit function." The "unit function" is generally written **1**, and it has been the subject of much discussion. When it is written as the right-hand member of an emf equation it merely means that unit potential difference is applied to the circuit in question at $t = 0$, the assumption being that the entire system is at rest at the time of closing the switch. The "unit function," **1**, is zero when $t < 0$, and equal to unity at $t = 0$ and $t > 0$.

A large number of Heaviside's electric circuit problems were carried out under the conditions of initial rest and unit voltage applied at $t = 0$. These requirements are sometimes called the Heaviside condition. It should be recognized, however, that with proper manipulation, operational methods can be employed when various other circuit conditions exist.

The Fundamental Time Operator.—p is employed as a symbol for differentiation with respect to the independent variable, time.

$$p = \frac{d}{dt}, \quad p^2 = \frac{d^2}{dt^2}, \quad \text{etc.}$$

Used as such, p may be manipulated algebraically in any ordinary differential equation. It becomes important to know the meaning of $1/p$. To have correct algebraic significance $1/p$ must be such that when it is multiplied by p the result is unity. We shall require that:

$$p \cdot \frac{1}{p} = 1$$

The above requirement is met if we let:

$$\frac{1}{p} = \int_0^t dt = t \qquad (1)$$

since

Likewise:

$$\frac{1}{p^2} = \frac{1}{p}\int_0^t dt = \frac{t^2}{\lfloor 2} \tag{2}$$

$$\frac{1}{p^3} = \frac{1}{p}\cdot\frac{1}{p}\cdot\int_0^t dt = \frac{t^3}{\lfloor 3} \tag{3}$$

etc.

The use of $1/p$ as an indefinite integral, $\int dt$, will also satisfy the algebraic requirement but it will not define $1/p$ uniquely, and such use should be conscientiously avoided in operational solutions. Otherwise boundary conditions may be violated and certain constants of integration may be lost. The correct use of $1/p$ cannot be overemphasized because its misuse will lead at once to mathematical inconsistencies. A careful analysis of the meanings of all p's which appear in the denominators of operational expressions will do much to reconcile operational terms with the more conventional types of functions.

The fundamental operator, $\frac{d}{dt}$, and the inverse operator, $\int_0^t dt$, are very common expressions in electric circuit equations. Inductive counter-voltages are of the form $L\left(\frac{d}{dt}\right)i$. Condenser counter-voltages, which result from the flow of current, take the form:

$$\frac{\int_0^t i\,dt}{C}$$

The indication of differentiation by p and the indication of $\int_0^t dt$ by $1/p$ in the fundamental circuit equations are the first steps in the operational method of attack.

The Direct Operational Method

Consider the equilibrium equation of the simple RLC series circuit to which unit voltage is applied at $t = 0$. Its operational form becomes:

$$\left(R + Lp + \frac{1}{pC}\right)i = 1$$

Equation (4) may be written in various ways, for example:

$$i = \frac{1}{\left(R + Lp + \frac{1}{pC}\right)} \mathbf{1} \tag{5}$$

$$i = \frac{p}{\left(Lp^2 + Rp + \frac{1}{C}\right)} \mathbf{1} \tag{6}$$

The interpretation of the function of p which operates on $\mathbf{1}$ is necessary if a solution for i is to be obtained. The direct operational method is to manipulate the function of p into a form that can be recognized and appreciated.

The only operational forms which have been defined up to this point are: p, p^2, etc., and $1/p$, $1/p^2$, etc. The method of attack is therefore indicated. Express the function of p in either ascending or descending powers of p if this is possible. Experience with operational forms will soon show that the function of p must, in general, be expanded in descending powers rather than in ascending powers of p if the complete solution for i is to be determined.

Heaviside set forth three rules for the evaluation of operational expressions, the first of which is given below.

The First Heaviside Rule.—If a function of p, say $\phi(p)$, can be expanded in a convergent series such that:

$$\phi(p) = a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} + \cdots \tag{7}$$

then:

$$\phi(p)\,\mathbf{1} = a_0 + a_1 t + \frac{a_2 t^2}{\lfloor 2} + \frac{a_3 t^3}{\lfloor 3} + \cdots \tag{8}$$

The proof of the above rule follows directly from the definitions which have been given for $1/p$, $1/p^2$, $1/p^3$, etc.

Writing the unit function symbol on the left-hand side of equation (8) is merely a warning that the function of p, $\phi(p)$, takes the form shown on the right-hand side of the equation only when it operates on a constant quantity, which in this case is unity.

Most of the functions of p which are encountered in elementary circuit theory may be evaluated by means of Heaviside's first rule.

Evaluation of Simple Functions.—*Example 1.* One of the most common functions of p is

$$\frac{1}{a + p}$$

where a is any positive constant.

$$\frac{1}{a+p}1 = \frac{1}{a}(1 - \epsilon^{-at}) \tag{9}$$

The above may be shown to be correct in terms of the definitions of $1/p$, $1/p^2$, etc.

Expanding $\frac{1}{a+p}$ in descending powers of p by ordinary algebraic division, the following is obtained:

$$\frac{1}{a+p} = \frac{1}{p} - \frac{a}{p^2} + \frac{a^2}{p^3} - \frac{a^3}{p^4} + \cdots \tag{10}$$

Substituting the values of $1/p$, $1/p^2$, $1/p^3$, etc., as defined by equations (1), (2), and (3), and recognizing that the operations are to be performed on constant quantities, the above expression takes the following form:

$$\frac{1}{a+p}1 = t - \frac{at^2}{\underline{|2}} + \frac{a^2t^3}{\underline{|3}} - \frac{a^3t^4}{\underline{|4}} + \cdots \tag{11}$$

An expansion of ϵ^{-at} by Maclaurin's series yields a somewhat similar expression. The details connected with the expansion of ϵ^{ax} are given on page 312.

$$\epsilon^{-at} = 1 - at + \frac{a^2t^2}{\underline{|2}} - \frac{a^3t^3}{\underline{|3}} + \frac{a^4t^4}{\underline{|4}} - \cdots \tag{12}$$

Rearranging:

$$\frac{1}{a}(1 - \epsilon^{-at}) = t - \frac{at^2}{\underline{|2}} + \frac{a^2t^3}{\underline{|3}} - \frac{a^3t^4}{\underline{|4}} + \cdots \tag{13}$$

Comparing equations (11) and (13) will show that:

$$\frac{1}{a+p}1 = \frac{1}{a}(1 - \epsilon^{-at})$$

Example 2.—Another very common function of p is:

$$\frac{p}{p+a}$$

$$\frac{p}{p+a}1 = \frac{1}{1+\frac{a}{p}}1 = \epsilon^{-at} \tag{14}$$

By division:

$$\frac{1}{1+\frac{a}{p}} = 1 - \frac{a}{p} + \frac{a^2}{p^2} - \frac{a^3}{p^3} + \cdots \tag{15}$$

$$\frac{1}{1+\frac{a}{p}}\mathbf{1} = 1 - at + \frac{a^2t^2}{\underline{|2}} - \frac{a^3t^3}{\underline{|3}} + \cdots \tag{16}$$

Likewise:

$$\epsilon^{-at} = 1 - at + \frac{a^2t^2}{\underline{|2}} - \frac{a^3t^3}{\underline{|3}} + \cdots \tag{17}$$

The relations stated in equation (14) are therefore established.

Example 3.—Let it be required to interpret the meaning of the following:

$$\frac{p}{a^2+p^2}\mathbf{1}$$

If the procedure as outlined in the above examples is carried through it will be found that:

$$\frac{p}{a^2+p^2}\mathbf{1} = \frac{1}{a}\sin at$$

The Use of Special Operators.—When functions of p can be evaluated in terms of ordinary functions they become useful operators. The evaluation of a given function of p in an operational solution is somewhat analogous to the evaluation of the integral in the conventional type of solution. Many operational forms have been evaluated, and tables of operational formulas may be found in almost any of the standard books which have been written on operational methods. These tables are to operational solutions what integral tables are to the ordinary solutions.

The operators which have been considered in the foregoing examples take the forms indicated provided the operations are performed on constant quantities. Obviously, if such operators are employed in connection with functions of the independent variable the full significance of the p's in the denominators can be realized only by performing the indicated integrations. For example:

$$\frac{E}{L(a+p)} = \frac{E}{L}\cdot\frac{1}{a+p}\mathbf{1} = \frac{E}{La}(1 - \epsilon^{-at}) \tag{18}$$

and

$$\frac{E}{R\left(1+\frac{a}{p}\right)} = \frac{E}{R}\cdot\frac{1}{1+\frac{a}{p}}1 = \frac{E}{R}\cdot\epsilon^{-at} \tag{19}$$

But

$$\frac{E_m \sin \omega t}{L(a+p)} \neq \frac{E_m \sin \omega t}{La}(1-\epsilon^{-at})$$

and

$$\frac{E_m \sin \omega t}{R\left(1+\frac{a}{p}\right)} \neq \frac{E_m \sin \omega t}{R}\epsilon^{-at}$$

A moment's consideration of the meaning of $1/p$, $1/p^2$, etc., that appear in the expanded forms of these operators will disclose the reasons for the above inequalities. The use of the differential and integral operators in connection with functions of the independent variable will be considered later.

Equation (18) illustrates the manner in which the operator $\frac{1}{a+p}$ is used to find the current rise in an RL circuit. In this case $a = R/L$. Equation (19) represents the current solution in an RC circuit to which a constant voltage, E, is applied at $t = 0$.

The general procedure to be followed in obtaining an operational solution of an ordinary differential equation may be summarized as follows:

1. Indicate differentiation with respect to the independent variable by means of the differential operator p. Indicate integration by means of $1/p$. It should be recognized that $1/p$ is used herein to denote a definite integration.
2. Manipulating p algebraically, solve explicitly for the dependent variable, if possible, in terms of p.
3. Interpret the solution in terms of known operators.

Boundary Conditions.—The manner in which the boundary conditions find their way into the final expressions constitutes the chief distinction between conventional and operational methods. In the former, the boundary conditions enter the final expressions by way of the constants of integration, whereas in the operational method those conditions are automatically implied or are imposed upon the original expressions. The use of $1/p$ as $\int_0^t dt$ eliminates the constants of inte-

gration but it automatically imposes the condition of initial rest upon the system.

When conditions other than initial rest are encountered it is necessary to write these facts into the original equations or to make certain circuit rearrangements which simulate the condition of rest at $t = 0$. Initial condenser charges may be written into the original operational expressions and treated thereafter as constant quantities. An example of circuit rearrangement to simulate initial rest is given in the operational solution of the RL decay current, page 57. The fact that the boundary conditions automatically enter into the operational solution gives it a certain advantage over the conventional solution where several constants of integration are involved.

Heaviside's Expansion Formula

Under the basic Heaviside conditions, namely, the circuit initially at rest and unit potential difference applied at $t = 0$, the explicit expression for current is of the form:

$$i = \frac{Y(p)}{Z(p)} \mathbf{1} \tag{20}$$

where $Y(p)$ is a function of p, and
$Z(p)$ is a function of p.

Equation (6) is a typical form. In this case $Y(p) = p$ and $Z(p) = Lp^2 + Rp + 1/C$. The highest power of p in $Y(p)$ will, in general, be lower than the highest of p in $Z(p)$. This condition need not be a limiting factor, however. If a constant potential difference of E volts is applied to the circuit at $t = 0$, the expression for i is:

$$i = E\frac{Y(p)}{Z(p)} \mathbf{1} \tag{21}$$

In the case of steady voltage applied, the result is affected only in a proportional manner. Manipulations of $Y(p)$ and $Z(p)$ together with the final interpretation of $\dfrac{Y(p)}{Z(p)}$ are not influenced by the presence of E in equation (21), since E is not a function of t. In general, $Z(p)$ takes the following form:

$$Z(p) = p^n + a_1p^{n-1} + a_2p^{n-2} + \tag{22}$$

It is assumed that n, $n - 1$, $n - 2$, etc., are real positive integers. Under these conditions, the polynomial form of $Z(p)$ may be factored.

$$\frac{Y(p)}{Z(p)} = \frac{Y(p)}{(p - p_1)(p - p_2)(p - p_3) \cdots} \tag{23}$$

where p_1, p_2, p_3, etc., are roots of $Z(p) = 0$.

If equation (23) is to be interpreted in terms of simple operators it is necessary to effect a separation of the right-hand member. Separation into partial fractions results in:

$$\frac{Y(p)}{Z(p)}1 = \frac{A}{p - p_1} + \frac{B}{p - p_2} + \frac{D}{p - p_3} + \cdots \tag{24}$$

from which:

$$Y(p) = A(p-p_2)(p-p_3) \cdots + B(p-p_1)(p-p_3) \cdots + \cdots \tag{25}$$

For $p = p_1$: $$A = \frac{Y(p_1)}{(p_1 - p_2)(p_1 - p_3) \cdots} \tag{26}$$

For $p = p_2$: $$B = \frac{Y(p_2)}{(p_2 - p_1)(p_2 - p_3) \cdots} \tag{27}$$

etc.

A, B, D, etc., are well-defined constants but they are of a rather unwieldy form. The denominators of equations (26) and (27) admit of simpler expression. Let:

$$Z(p) = (p - p_1)F(p)$$

where $F(p)$ is $(p - p_2)(p - p_3) \cdots$.

It follows directly that

$$\frac{dZ(p)}{dp} = (p - p_1)\frac{dF(p)}{dp} + F(p)$$

Therefore:

$$\left[\frac{dZ(p)}{dp}\right]_{(\text{for } p = p_1)} = (p_1 - p_2)(p_1 - p_3) \cdots \tag{28}$$

In a similar manner it may be shown that:

$$\left[\frac{dZ(p)}{dp}\right]_{(\text{for } p = p_2)} = (p_2 - p_1)(p_2 - p_3) \cdots \tag{29}$$

The expressions given above not only simplify the writing of A, B, D,

etc., in terms of p_1, p_2, p_3, etc., but often simplify the actual evaluation of those constants.

$$A = \left[\frac{Y(p)}{\dfrac{dZ(p)}{dp}}\right]_{\text{for } p=p_1} \tag{30}$$

$$B = \left[\frac{Y(p)}{\dfrac{dZ(p)}{dp}}\right]_{\text{for } p=p_2} \tag{31}$$

etc.

It has been shown that:

$$\frac{1}{p+a}1 = \frac{1}{a}(1 - \epsilon^{-at}).$$

Therefore:

$$\frac{1}{p-p_1}1 = -\frac{1}{p_1}(1 - \epsilon^{p_1t}) \tag{32}$$

$$\frac{1}{p-p_2}1 = -\frac{1}{p_2}(1 - \epsilon^{p_2t}) \tag{33}$$

etc.

In the ordinary dissipative type of circuit p_1 and p_2 are either real negative numbers or complex numbers with real parts which are negative. The cases where p_1, p_2, etc., are positive reals or contain positive real components should be given special consideration.[5] If p_1 is inherently a negative number it reduces the operator $\dfrac{1}{p-p_1}$ to the form $\dfrac{1}{p+a}$. Expressing equation (24) in terms of the above equalities:

$$\frac{Y(p)}{Z(p)}1 = \left[-\frac{A}{p_1}-\frac{B}{p_2}-\frac{D}{p_3}-\cdots\right]+\left[\frac{A\epsilon^{p_1t}}{p_1}+\frac{B\epsilon^{p_2t}}{p_2}+\frac{D\epsilon^{p_3t}}{p_3}+\cdots\right] \tag{34}$$

Reference to equation (24) will show that:

$$\left[-\frac{A}{p_1}-\frac{B}{p_2}-\frac{D}{p_3}-\cdots\right] = \left[\frac{Y(p)}{Z(p)}\right]_{(\text{for } p=0)} = \frac{Y(0)}{Z(0)}$$

The remainder of equation (34) consists of the sum of as many exponen-

[5] Bush, "Operational Circuit Analysis," Chapter XIII.

tial terms as $Z(p) = 0$ has roots. The complete expression for i may now be written in the following form:

$$i = E\left[\frac{Y(0)}{Z(0)} + \sum_p \frac{Y(p)\,\epsilon^{pt}}{p\,\dfrac{dZ(p)}{dp}}\right] \tag{35}$$

$$\begin{aligned} p &= p_1 \\ &= p_2 \\ &= p_3 \\ &\text{etc.} \end{aligned}$$

The manner in which equation (35) has been developed will indicate its limitations.

1. Initial rest and the application of constant potential difference to the circuit have been implied. It may be noted in passing, however, that with suitable transformations, the equation is capable of extension to alternating currents and potentials.

2. $\dfrac{Y(p)}{Z(p)}$ has been treated as a rational fraction. If it is not in that form originally, and $Y(p)$ and $Z(p)$ are of rational polynominal form, $Y(p)$ may, by division, be made of lower degree than $Z(p)$. The condition of irrationality is of more theoretical than practical interest in an elementary treatment of the subject. In more advanced treatments, fractional exponents of p are considered and certain irrational forms have been given mathematical and physical interpretation.

3. $Z(p)$ has been broken up into distinct linear factors, none of which is repeated. Special consideration must be given to those singular cases in which $[Z(p) = 0]$ has equal roots.

Equation (35), known as Heaviside's expansion formula, appears in various forms throughout the literature. Its derivation is based merely upon the interpretation of the meaning of a series of operators of the form $\dfrac{1}{p+a}$. The neatness and compactness of the final form are worthy of attention. Such a concise and explicit expression of the dependent variable, void of undefined constants of integration, is a powerful instrument of analysis. After the evaluation of the roots [6] of $[Z(p) = 0]$, a particular solution is at once obtainable. However, the beginner should not expect that the actual labor involved in obtaining a numerical solution is greatly reduced over the amount of labor

[6] See Section III of the Appendix for a method of evaluating the roots of high degree algebraic equations.

required by other methods. Such an expectancy will lead to a decided disappointment when, for example, he attempts to evaluate $\dfrac{Y(p)}{p\left[\dfrac{dZ(p)}{dp}\right]}$ for a series of complex roots. In many cases, the systematized procedure that is outlined by the formula will reduce the labor involved to a minimum. The necessity of separately determining the constants of integration is eliminated, and this in itself will often justify the use of Heaviside's expansion formula. Analytical and numerical examples of its application are given in Chapters IV and V of the text.

Although the second Heaviside rule, as the expansion theorem is sometimes called, simplifies the solutions of certain elementary problems it finds its greatest field of usefulness in problems involving partial differential equations.

Use of Operators in Connection with sin $(\omega t + \lambda)$

When used in connection with a function of time, operators such as those shown in equations (9) and (14) yield results which are entirely different from those obtained when the operations are performed upon constant quantities. This is to be expected, in view of the meanings which have been attached to p and $1/p$. A function of time upon which operations are frequently made is:

$$f(t) = \sin(\omega t + \lambda), \text{ or its equivalent,}$$

where ω and λ are constants.

Example 1.—Let it be required to interpret the meaning of:

$$\frac{1}{p + a}\sin(\omega t + \lambda) \tag{36}$$

There are several methods of attack, most of which require a rather advanced mathematical background for a true appreciation of the result. A more elementary, and somewhat cumbersome, method is outlined below. Heaviside, however, actually employed this elementary method in certain instances.[7]

Expanding in descending powers of p:

$$\frac{1}{p + a} = \frac{1}{p} - \frac{a}{p^2} + \frac{a^2}{p^3} - \frac{a^3}{p^4} + \cdots \tag{37}$$

[7] Heaviside, "Electromagnetic Theory," Vol. II, pages 132–133.

The expression in (36) may now be written:

$$\left[\frac{1}{p} - \frac{a}{p^2} + \frac{a^2}{p^3} - \frac{a^3}{p^4} + \cdots\right] \sin(\omega t + \lambda) \tag{38}$$

A series of operations is indicated.

$$\frac{1}{p}\sin(\omega t + \lambda) = \int_0^t \sin(\omega t + \lambda)dt = \frac{1}{\omega}[-\cos(\omega t + \lambda) + \cos\lambda] \tag{39}$$

$$-\frac{a}{p^2}\sin(\omega t + \lambda) = -\frac{a}{\omega^2}[-\sin(\omega t + \lambda) + \sin\lambda + \omega t\cos\lambda] \tag{40}$$

$$\frac{a^2}{p^3}\sin(\omega t + \lambda) = \frac{a^2}{\omega^3}\left[\cos(\omega t + \lambda) - \cos\lambda + \omega t\sin\lambda + \frac{\omega^2 t^2}{\lfloor 2}\cos\lambda\right] \tag{41}$$

$$-\frac{a^3}{p^4}\sin(\omega t + \lambda) = -\frac{a^3}{\omega^4}\left[\sin(\omega t + \lambda) - \sin\lambda - \omega t\cos\lambda + \frac{\omega^2 t^2}{\lfloor 2}\sin\lambda + \frac{\omega^3 t^3}{\lfloor 3}\cos\lambda\right] \tag{42}$$

etc.

Even though the process of integration goes on indefinitely, only the first few integrations are required to predict the summation of all the terms that will appear. The uniformity with which the $\sin(\omega t + \lambda)$ term alternates with the $\cos(\omega t + \lambda)$ term and the fact that there is a definite ratio between their coefficients makes the summation of all those terms a simple matter.

Summing the $\sin(\omega t + \lambda)$ and $\cos(\omega t + \lambda)$ terms of (39), (40), (41), (42), etc., the following collections are obtained:

$$\sin(\omega t + \lambda)\cdot\frac{a}{\omega^2}\left[1 - \frac{a^2}{\omega^2} + \frac{a^4}{\omega^4} - \cdots\right] \tag{43}$$

and

$$-\cos(\omega t + \lambda)\cdot\frac{1}{\omega}\left[1 - \frac{a^2}{\omega^2} + \frac{a^4}{\omega^4} - \cdots\right] \tag{44}$$

The sum of all the $\sin(\omega t + \lambda)$ and $\cos(\omega t + \lambda)$ terms is:

$$\frac{1}{\sqrt{a^2 + \omega^2}}\sin(\omega t + \lambda - \theta) \tag{45}$$

where $\theta = \tan^{-1}\omega/a$.

In combining (43) and (44) it will be recognized that the bracket terms are equal to $\dfrac{\omega^2}{a^2 + \omega^2}$.

In addition to the terms contained in (45) there are other infinite

series contained in (39), (40), (41), (42), etc. Following the $\cos\lambda$ term that appears in equation (39) through the successive integrations and adding them yield:

$$\cos\lambda\left(\frac{1}{\omega}-\frac{at}{\omega}+\frac{a^2t^2}{\omega\,\underline{|2}}-\frac{a^3t^3}{\omega\,\underline{|3}}+\cdots\right)=\frac{\epsilon^{-at}}{\omega}\cos\lambda$$

But other $\cos\lambda$ terms continue to appear. Taking them all into account the sum of all the $\cos\lambda$ terms is:

$$\frac{\epsilon^{-at}}{\omega}\cos\lambda\left[1-\frac{a^2}{\omega^2}+\frac{a^4}{\omega^4}+\cdots\right]=\frac{\epsilon^{-at}}{\sqrt{a^2+\omega^2}}\cos\lambda\sin\theta \tag{46}$$

where θ is again $\tan^{-1}\omega/a$.

A similar summation of all the $\sin\lambda$ terms that appear throughout the infinite number of integrations is:

$$-\frac{a\epsilon^{-at}}{\omega^2}\sin\lambda\left[1-\frac{a^2}{\omega^2}+\frac{a^4}{\omega^4}-\cdots\right]=\frac{-\epsilon^{-at}}{\sqrt{a^2+\omega^2}}\sin\lambda\cos\theta \tag{47}$$

Combining (45), (46), and (47) shows the result of operating on $\sin(\omega t+\lambda)$ with $\dfrac{1}{a+p}$. The net result is:

$$\frac{1}{\sqrt{a^2+\omega^2}}\left[\sin(\omega t+\lambda-\theta)-\epsilon^{-at}\sin(\lambda-\theta)\right] \tag{48}$$

The above expression may be used to effect operational solutions when the applied voltage varies sinusoidally and when

$$\frac{Y(p)}{Z(p)}=\frac{1}{a+p}$$

For an illustration of its application see equations (18), (19), (20), and (21) of Chapter VII. These equations comprise the current solution in the RL circuit to which sinusoidal voltage is suddenly applied.

Example 2.—Let it be required to interpret the meaning of:

$$\frac{p}{p+a}\sin(\omega t+\lambda) \tag{49}$$

Expanding in descending powers of p·

$$\frac{p}{p+a}=1-\frac{a}{p}+\frac{a^2}{p^2}-\frac{a^3}{p^3}+\cdots \tag{50}$$

Expression (49) may be written:

$$\left[1-\frac{a}{p}+\frac{a^2}{p^2}-\frac{a^3}{p^3}+\cdots\right]\sin(\omega t+\lambda) \tag{51}$$

Performing the indicated operations and numbering them as they are performed from left to right:

(1) $$\sin(\omega t+\lambda)=\sin(\omega t+\lambda) \tag{52}$$

(2) $$-\frac{a}{p}\sin(\omega t+\lambda)=-\frac{a}{\omega}[-\cos(\omega t+\lambda)+\cos\lambda] \tag{53}$$

(3) $$\frac{a^2}{p^2}\sin(\omega t+\lambda)=\frac{a^2}{\omega^2}[-\sin(\omega t+\lambda)+\sin\lambda+\omega t\cos\lambda] \tag{54}$$

(4) $$-\frac{a^3}{p^3}\sin(\omega t+\lambda)$$
$$=-\frac{a^3}{\omega^3}\left[\cos(\omega t+\lambda)-\cos\lambda+\omega t\sin\lambda+\frac{\omega^2t^2}{\lfloor 2}\cos\lambda\right] \tag{55}$$

(5) $$\frac{a^4}{p^4}\sin(\omega t+\lambda)$$
$$=\frac{a^4}{\omega^4}\left[\sin(\omega t+\lambda)-\sin\lambda-\omega t\cos\lambda+\frac{\omega^2t^2}{\lfloor 2}\sin\lambda+\frac{\omega^3t^3}{\lfloor 3}\cos\lambda\right] \tag{56}$$

etc.

The sum of all the [sin $(\omega t+\lambda)$] terms is:

$$\sin(\omega t+\lambda)\left[1-\frac{a^2}{\omega^2}+\frac{a^4}{\omega^4}-\cdots\right]=\frac{\omega^2}{a^2+\omega^2}\sin(\omega t+\lambda) \tag{57}$$

The sum of all the [cos $(\omega t+\lambda)$] terms is:

$$\cos(\omega t+\lambda)\left[\frac{a}{\omega}-\frac{a^3}{\omega^3}+\frac{a^5}{\omega^5}-\cdots\right]=\frac{a\omega}{a^2+\omega^2}\cos(\omega t+\lambda) \tag{58}$$

Combining (57) and (58), the sum of all the [sin $(\omega t+\lambda)$] terms and the [cos $(\omega t+\lambda)$] terms becomes:

$$\omega\left[\frac{\omega}{a^2+\omega^2}\sin(\omega t+\lambda)+\frac{a}{a^2+\omega^2}\cos(\omega t+\lambda)\right]$$
$$=\frac{\omega}{\sqrt{a^2+\omega^2}}\sin(\omega t+\lambda-\theta) \tag{59}$$

where $\theta=\tan^{-1}-a/\omega$.

It remains to collect all the ($\cos\lambda$) and ($\sin\lambda$) terms in (53), (54), (55), (56), etc. A ($\cos\lambda$) term first appears in equation (53). This particular term is successively integrated in (54), (55), (56), etc. The sum of the first ($\cos\lambda$) term and all the successive integrations of that term is:

$$\cos\lambda\left[-\frac{a}{\omega}+\frac{a^2t}{\omega}-\frac{a^3t^2}{\omega\lfloor 2}+\frac{a^4t^3}{\omega\lfloor 3}-\cdots\right]=-\frac{a}{\omega}\cos\lambda\cdot\epsilon^{-at} \qquad (60)$$

A second ($\cos\lambda$) term appears in (55), namely:

$$\frac{a^3}{\omega^3}\cos\lambda$$

The sum of the second ($\cos\lambda$) term and all the successive integrations upon that term is:

$$\frac{a^3}{\omega^3}\cos\lambda\left[1-at+\frac{a^2t^2}{\lfloor 2}-\cdots\right]=\frac{a^3}{\omega^3}\cos\lambda\cdot\epsilon^{-at} \qquad (61)$$

$$\Sigma\ (\cos\lambda)\text{ terms} = \epsilon^{-at}\cos\lambda\left[-\frac{a}{\omega}+\frac{a^3}{\omega^3}-\frac{a^5}{\omega^5}+\cdots\right] \qquad (62)$$

$$= -\frac{a\omega}{a^2+\omega^2}\cos\lambda\epsilon^{-at} \qquad (63)$$

In a similar manner, all the ($\sin\lambda$) terms that appear in (54), (55), (56), etc., may be collected.

$$\Sigma\ (\sin\lambda)\text{ terms} = \frac{a^2}{a^2+\omega^2}\sin\lambda\ \epsilon^{-at} \qquad (64)$$

Combining (63) and (64):

$$\left[\frac{a}{a^2+\omega^2}\sin\lambda-\frac{a\omega}{a^2+\omega^2}\cos\lambda\right]\epsilon^{-at}=\frac{-a}{\sqrt{a^2+\omega^2}}\cos(\lambda-\theta)\epsilon^{-at} \qquad (65)$$

$$= \sin\theta\cos(\lambda-\theta)\epsilon^{-at} \qquad (66)$$

The sum of (59) and (66) is, therefore, the net result of operating with $\frac{p}{p+a}$ upon $\sin(\omega t+\lambda)$.

$$\frac{p}{p+a}\sin(\omega t+\lambda)=\frac{\omega}{\sqrt{a^2+\omega^2}}\sin(\omega t+\lambda-\theta)+\sin\theta\cos(\lambda-\theta)\epsilon^{-at} \qquad (67)$$

where $\theta = \tan^{-1}\frac{-a}{\omega}$.

In a certain class of circuit problems the first term of the right-hand member of equation (67) represents the steady-state component of the complete solution, and the second term represents the transient component. The operation indicated in (67) is used in the solution of RC circuits to which a sinusoidal voltage is applied at $t = 0$. An illustration of its application may be found under the mathematical analysis of the RC circuit in Chapter VII.

Special Devices

Duhamel's Integral.—The examples given above illustrate the manner in which certain important operational forms may be interpreted in terms of the basic definitions of p and $1/p$. But it is necessary to follow through only one or two elementary evaluations of the kind given to fully appreciate the need for a more refined and systematic approach.

Various forms of Duhamel's integral have been used successfully in the evaluation of operational expressions. The reader is referred to Bush's "Operational Circuit Analysis," Chapter V, or to Berg's "Heaviside's Operational Calculus," Chapter XIV, for an explanation of Duhamel's integral as applied to electric circuit analysis.

The Extended Expansion Formula.—Expressions which are comparable to equation (35) have been worked out for alternating currents and voltages. The first term on the right-hand side of the equation represents the steady-state solution, and the second term represents the transient solution in the a-c circuit. It is questionable whether the expansion formula thus extended finds any place in elementary circuit analysis. The notation which is required to signify the general summation of all the various components of alternating current is somewhat involved, and the actual labor required to effect a solution in a particular case is very often a serious handicap.

The Third Heaviside Rule.—A statement of the third Heaviside rule is that if a function of p, say $\phi(p)$, can be written as follows:

$$\phi(p) = (a_0 + a_1 p + a_2 p^2 + \ldots) + p^{1/2}(b_0 + b_1 p + b_2 p^2 + \ldots) \qquad (68)$$

then:

$$\phi(p)\mathbf{1} = a_0 + \frac{1}{\sqrt{\pi t}}\left(b_0 - \frac{b_1}{2t} + \frac{b_2 \cdot 1 \cdot 3}{(2t)^2} - \frac{b_3 \cdot 1 \cdot 3 \cdot 5}{(2t)^3} + \ldots\right) \qquad (69)$$

asymptotically. This rule or theorem is very important and very useful in circuit problems which involve distributed circuit parameters.

Inasmuch as the third rule is not used in the text its proof will not

be attempted. Moreover, a systematic and rigorous proof may become somewhat involved mathematically.

Heaviside evaluated many operational forms wherein the basic operation is:

$$p^{1/2}1 = (\pi t)^{-1/2} \tag{70}$$

In terms of the definition that has been given for p, the left-hand member of the above expression signifies fractional differentiation, and such it is. Heaviside used equation (70) as a " fundamental formula " in many types of problems.[8]

[8] Heaviside, "Electromagnetic Theory," Vol. II.

SECTION III

THE GRAEFFE METHOD OF SOLVING ALGEBRAIC EQUATIONS

General.—It is desirable that the student of electrical circuit theory acquaint himself with some method of solving high degree algebraic equations, the roots of which may be real or complex. A most effective method of solving linear differential equations of the higher orders having constant coefficients involves their reduction to algebraic form. The evaluation of the roots of the algebraic equation is an essential step in the actual solution of the problem. The Graeffe method of solving algebraic equations is briefly described in the following paragraphs. Emphasis is placed on the actual method of procedure that is followed in evaluating the roots of particular equations found in this text.[1]

The first step in the Graeffe method of solution is the establishment of a new equation whose roots are the squares of the roots of the original equation. Given the numerical coefficients of an algebraic equation, it is a relatively simple matter to form an equation whose roots are the squares of the roots of the first equation. For example, let the original equation be:

$$x^3 + a_1x^2 + a_2x + a_3 = 0 \tag{1}$$

The numerical coefficients and the constant term are represented by a_1, a_2, and a_3.

Let the unknown roots of the equation be designated as $-a$, $-b$, and $-c$. Uniformity of signs in succeeding equations makes it desirable to employ the minus signs in connection with the roots. The underlying principle of the Graeffe method does not concern itself with the signs of the roots.

Equation (1) may now be written:

$$(x + a)(x + b)(x + c) = 0 \tag{2}$$

or

$$x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc = 0 \tag{3}$$

[1] Treatments of the Graeffe method which are more generally applicable may be found in: Whittaker and Robinson, "The Calculus of Observations." Berg, "Heaviside's Operational Calculus." Runge, "Numerisches Rechnen."

It will be observed that:

$$(a + b + c) = a_1,\ (ab + bc + ca) = a_2,\ abc = a_3$$

a_1, a_2, and a_3 are, presumably, known.

If $a >>> b >>> c$:

$$a = a_1 \text{ approximately, in magnitude} \tag{4}$$

$$ab = a_2 \text{ approximately, in magnitude} \tag{5}$$

and

$$abc = a_3 \text{ in magnitude} \tag{6}$$

Assuming that $a \neq b \neq c$, it is possible by means of the root-squaring process to arrive at an equation whose roots, a^m, b^m, and c^m differ by such vast amounts that b^m is negligibly small as compared with a^m, and c^m is negligibly small as compared with b^m. Under these conditions the relationships stated in (4), (5), and (6) may be employed to find a^m, b^m, and c^m. Since m is a known power, the values of a, b, and c follow directly. The important feature of the method is the determination of a^m, b^m, and c^m by means of the root-squaring process.

A general statement of the root-squaring process may be made as follows:

Given the equation:

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-4} + \ldots + a_n = 0$$

The equation whose roots are the squares (in magnitude)[2] of the roots of the above equation is:

$$x^n + \left.\begin{matrix} a_1^2 \\ -2a_2 \end{matrix}\right\} x^{n-1} + \left.\begin{matrix} a_2^2 \\ -2a_1a_3 \\ +2a_4 \end{matrix}\right\} x^{n-2} + \left.\begin{matrix} a_3^2 \\ -2a_2a_4 \\ +2a_1a_5 \\ -2a_6 \end{matrix}\right\} x^{n-3} + \left.\begin{matrix} a_4^2 \\ -2a_3a_5 \\ +2a_2a_6 \\ -2a_1a_7 \\ +2a_8 \end{matrix}\right\} x^{n-4} + \ldots = 0$$

[2] The roots of the original equation are considered to be $-a$, $-b$, $-c$, $-d$, etc. The roots of the newly formed equation are $-a^2$, $-b^2$, $-c^2$, $-d^2$, etc. The roots of an equation with their signs reversed are sometimes called Encke roots. With such designation, the newly formed equation is the equation whose Encke roots are the squares of the Encke roots of the original equation. This particular distinction regarding the signs of the roots in the newly formed equation and the original equation is tacitly assumed throughout the present discussion. When the statement is made that the roots of equation one are the squares of the roots of equation two, it is meant that the magnitudes of the roots of equation one are the squares of the magnitudes of the roots of equation two without regard to the signs of the quantities involved.

The coefficient of a given term in the equation whose roots are the squares of the roots of the original equation may be determined in the following manner. Square the coefficient of the given term in the original equation and add, with alternate minus and plus signs, the doubled products of the coefficients of the terms equally spaced from the given term. For example, the equation whose roots are the squares of the roots of equation (2) is:

$$(x + a^2)(x + b^2)(x + c^2) = 0 \tag{7}$$

or:

$$x^3 + (a^2 + b^2 + c^2)x^2 + (a^2b^2 + b^2c^2 + c^2a^2)x + a^2b^2c^2 = 0 \tag{8}$$

The coefficient of the x^2 term in the above equation is the square of the coefficient of the x^2 term in equation (3) minus two times the product of the coefficients of the x^3 term and the x term. In equation form:

$$(a^2 + b^2 + c^2) = (a + b + c)^2 - 2\,[1 \cdot (ab + bc + ca)] \tag{9}$$

There is only one doubled product term in this coefficient since there is no term which is two removed from the x^2 term to the left. In the case of absent terms their coefficients may be considered to be zero.

The coefficient of the x term is formed in a similar manner.

$$(a^2b^2 + b^2c^2 + c^2a^2) = (ab + bc + ca)^2 - 2\,[(a + b + c)(abc)] \tag{10}$$

The coefficient of the x^3 term in the new equation is merely unity since it is unity in the original equation and there is no doubled product term, there being no term to the left of x^3. The constant term in the new equation is, obviously, $a^2b^2c^2$.

If the original equation had been of higher degree than the third the same general procedure could have been employed in finding the coefficients of the equation whose roots are the squares of the roots of the original equation. For the sake of illustration let the original equation be:

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{11}$$

or:

$$(x + a)(x + b)(x + c)(x + d) = 0 \tag{12}$$

where the roots are $-a$, $-b$, $-c$, and $-d$.

Performing the operations indicated in (12) the equation takes the form:

$$x^4 + (a + b + c + d)x^3 + (ab + bc + ca + ad + bd + cd)x^2$$
$$+ (abc + bcd + cda + dab)x + abcd = 0 \tag{13}$$

The equation whose roots are the squares of the roots of equation (12) is:

$$x^4 + (a^2 + b^2 + c^2 + d^2)x^3 + (a^2b^2 + b^2c^2 + c^2a^2 + a^2d^2 + b^2d^2 + c^2d^2)x^2 + (a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2)x + a^2b^2c^2d^2 = 0 \qquad (14)$$

The coefficient of the x^3 term in the above equation is:

$$(a + b + c + d)^2 - 2\,[1 \cdot (ab + bc + ca + ad + bd + cd)]$$

The above expression is the square of the x^3 term in the original equation minus two times the product of the coefficients of the x^4 and the x^2 term.

A detailed study of the coefficient of the x^2 term in equation (14) will show that it is equivalent to:

$$(ab + bc + ca + ad + bd + cd)^2 - 2\,[(a + b + c + d)(abc + bcd + cda + dab)] + 2\,[1 \cdot (abcd)]$$

The above expression is the square of the x^2 term in the original equation, minus two times the product of the coefficients of the x^3 term and the x term, plus two times the product of the coefficient of the x^4 term and the constant term.

Although the particular examples given above do not constitute a generalized proof of the root-squaring process they do illustrate the manner in which the coefficients of the new equation are formed. The equation whose roots are the squares of the roots of the "squared" equation may, of course, be formed in the manner outlined above, the "squared" equation being treated as though it were the original equation. The roots of the equation thus formed are the fourth power of the roots of the first equation, and if the roots are originally of different magnitudes, much greater differences will now exist between the magnitudes of a^4, b^4, c^4, etc. By continuing the root-squaring process the value of b^m may be made insignificantly small as compared with a^m, and c^m insignificantly small as compared with b^m. It has been assumed that $a > b > c$, and that their magnitudes are greater than unity.

Of course, it is important to be able to recognize the point at which the root-squaring process can be discontinued. This point is indicated by the magnitude of the squares of the coefficients of the preceding equation relative to the magnitude of the doubled product terms. If all the roots of the equation are real and different all the doubled product terms will eventually become negligibly small as compared with the squared terms. Equation (9) will serve as an illustration. If $a >>> b >>> c$ the doubled product term on the right-hand side

of the equation is very much smaller than the squared term, and if the differences between the roots are sufficiently great the doubled product term may, as an approximation, be neglected. When the doubled product terms become negligibly small the root-squaring process may be discontinued and the roots evaluated. Under these conditions:

$$a^m >>> b^m >>> c^m >>> d^m \quad \text{etc.}$$

In a third degree equation:

$a^m = a_1$ of the mth equation to a close degree of accuracy,

$a^m b^m = a_2$ of the mth equation to a close degree of accuracy, and

$a^m b^m c^m = a_3$ of the mth equation.

The magnitudes of the roots of the original equation may be found from the above relationships. It is somewhat more convenient to recognize that $abc = a_3$ of the original equation in the evaluation of c than to employ the larger figures of the mth equation. The correct signs of the roots are determined by substitution in the original equation. Actual evaluation of the numerical roots of different types of equations is a most satisfactory means of becoming acquainted with the details of the method.

Solution of a Third Degree Equation—Real Roots

Let it be required to find the roots of the following equation by the root-squaring method:

$$x^3 + 5x^2 - 22x - 56 = 0 \tag{15}$$

The detailed procedure of the root-squaring process is best carried out in tabular form as shown below. The coefficients of the new equations are formed in accordance with the rules previously stated.

It will be observed that the doubled product terms encountered in building up the coefficients of the "32nd power equation" are negligibly small as compared with the squared terms. The coefficients of the "32nd power equation" are sensibly the squares of the corresponding coefficients of the "16th power equation," and if the root-squaring process is continued the coefficients of the new equation will continue to be the squares of those in the preceding equation. As shown previously, the coefficient of the x^2 term in a third-degree equation whose roots are the 32nd power of the roots of the original equation is $(a^{32} + b^{32} + c^{32})$. The fact that the doubled product terms have

Term	x^3	x^2	x	Constant
Coefficients of original equation..	1	5	-22	-56
Squares of above coefficients.. . .	1	25	484	3136
Doubled products...		44	560	
Coefficients of 2nd power equation...	1	69	1044	3136
Squares of above coefficients......	1	4761	109×10^4	983×10^4
Doubled products........... .. .		-2088	$-43\ 3\times10^4$	
Coefficients of 4th power equation .	1	2673	65.7×10^4	983×10^4
Squares of above coefficients	1	$714\ 5\times10^4$	4316×10^8	9663×10^{10}
Doubled products.........		$-131\ 4\times10^4$	-526×10^8	
Coefficients of 8th power equation .	1	$583\ 1\times10^4$	3790×10^8	9663×10^{10}
Squares of above coefficients . ..	1	3400×10^{10}	1436×10^{20}	9337×10^{24}
Doubled products.......		-76×10^{10}	-11×10^{20}	
Coefficients of 16th power equation	1	3324×10^{10}	1425×10^{20}	9337×10^{24}
Squares of above coefficients.......	1	1105×10^{24}	2031×10^{43}	8718×10^{52}
Doubled products...............		-0.3×10^{24}	$-0\ 06\times10^{43}$	
Coefficients of 32nd power equation.	1	1105×10^{24}	2031×10^{43}	8718×10^{52}

become negligible indicates that b^{32} and c^{32} are insignificantly small as compared with a^{32}.

Therefore:

$$a^{32} = 1105 \times 10^{24}$$

from which:

$$\log_{10} a = \frac{27.04336}{32} = 0.84510$$

and

$$a = 7.000$$

Substitution shown that -7 satisfies the original equation; hence, the numerically largest of the roots is -7.

Since the Graeffe method is inherently only an approximate method of solution, the degree of accuracy attained is an important consideration. In the present example the numerical work in the root-squaring process has been carried out to four significant figures, and under these conditions the root, -7, has been evaluated to a high degree of accuracy. The error involved is not greater than one part in 7000. If slide-rule accuracy had been employed in the root-squaring process a^{32} would probably have been $1105 \times 10^{24} \pm 2$ per cent. If the error introduced by the slide rule is plus 2 per cent:

$$a^{32} = 1126 \times 10^{24}$$

$$a = 7.004$$

In this particular case the use of the slide rule in the root-squaring process would have resulted in an error of less than 1/10th per cent. In general the use of a slide rule is permissible in building up the coefficients of the equation whose roots are some high power of the roots of the original equation. The remaining steps in the actual evaluation of the roots should be carried out to one or two more significant figures than can usually be obtained on the slide rule. The use of five-place logarithms together with the corresponding number of significant places is recommended in the later stages of the root evaluation.

The second largest root of the third-degree equation which is under consideration may be evaluated in the following manner:

$$a^{32}b^{32} = 2031 \times 10^{43}$$

$$b^{32} = \frac{2031 \times 10^{43}}{1105 \times 10^{24}}$$

$$b^{32} = 1.838 \times 10^{19}$$

$$b = 3.999, \text{ say } 4.00$$

Substitution in the original equation shows that plus 4 is one of the roots.

Since two of the roots have been evaluated, the constant term in the original equation may be employed in finding the remaining root.

$$abc = 56$$

$$c = \tfrac{56}{28} = 2$$

Negative 2 satisfies equation (15), the roots of which are -7, $+4$, and -2.

General Solution of a Third-degree Equation. One Pair of Complex Roots

The presence of complex roots is indicated by the persistence of the doubled product terms in certain of the numerical columns that are formed as the root-squaring process is carried out. A detailed study of the coefficients of an equation with complex roots will reveal the reason for the failure of the doubled product terms to disappear. Suppose that the roots of the equation are:

(1) $-(u + jv) = -r\epsilon^{+j\phi}$

(2) $-(u - jv) = -r\epsilon^{-j\phi}$

(3) $-a$, the real root

In factored form the third-degree equation may be written as:

$$(x + r\epsilon^{j\phi})(x + r\epsilon^{-j\phi})(x + a) = 0 \tag{16}$$

An equivalent form of the above equation is:

$$(x + u + jv)(x + u - jv)(x + a) = 0 \tag{17}$$

Performing the operations indicated in (16) and reducing the exponential terms to their trigonometric equivalent, the equation becomes

$$x^3 + (2r \cos \phi + a)x^2 + (2ar \cos \phi + r^2)x + ar^2 = 0 \tag{18}$$

The equation whose roots are the squares (in magnitude) of the roots of the above equation is:

$$x^3 + (2r^2 \cos 2\phi + a^2)x^2 + (2a^2r^2 \cos 2\phi + r^4)x + a^2r^4 = 0 \tag{19}$$

If the root-squaring process is continued until the roots become the mth power of the original roots the equation takes the following form:

$$x^3 + (2r^m \cos m\phi + a^m)x^2 + (2a^mr^m \cos m\phi + r^{2m})\,x + a^mr^{2m} = 0 \tag{20}$$

When $r > a$.—If r is greater than a the predominant factor in the coefficient of the x^2 term is $(2r^m \cos m\phi)$. As the root-squaring process is being carried out this term may acquire alternate plus and minus signs, depending upon the values of m and ϕ. The coefficient of the x^2 term $(2r^m \cos m\phi + a^m)$, will never reach a point in the root-squaring process such that it becomes successively the square of the coefficient of the preceding equation. If, therefore, the numerical coefficients of the x^2 terms oscillate in sign and the doubled product terms remain significantly large as compared with the squared terms throughout the root-squaring process, the existence of one pair of complex roots is

indicated. Furthermore, the fact that this occurs in the x^2 column indicates that the magnitude of the complex root is greater than the magnitude of the real root. The coefficient of the x^2 term, even in the "mth power equation," cannot be employed in the direct evaluation of the magnitude of r since the value of ϕ is presumably unknown at this point. Other coefficients of the "mth power equation" must be employed in the evaluation of r and a.

The dominant factor in the coefficient of the x term is r^{2m} if $r > a$. Inspection of the coefficient of the x term in equation (20) will show that it is of such a nature that, in the root-squaring process, a point will eventually be reached after which the coefficient in the new equation becomes the square of the coefficient in the preceding equation. For such to be the case the doubled product terms must become negligibly small as compared with the squared terms. The root-squaring process may, therefore, be discontinued when the doubled products in the coefficients of the x term become insignificantly small. The magnitudes of r and a may then be evaluated inasmuch as two very simple relationships can be found. Under the conditions stated:

$$r^{2m} = \text{coefficient of the } x \text{ term in the "}m\text{th power equation"}$$

and:

$$a^m r^{2m} = \text{constant term of the "}m\text{th power equation"}$$

When $a > r$.—a^m is the dominant factor in the coefficient of the x^2 term if a $> r$. Under these conditions a point is reached in the root-squaring process after which the coefficient of the x^2 term in the newly formed equation becomes the square of the coefficient of the x^2 term in the preceding equation. The doubled product terms become negligibly small as in the case of all real roots.

The existence of the complex roots is indicated by the oscillation in sign of the coefficients of the x term and the persistence of the doubled product terms in the successively formed coefficients of the x term as the root-squaring process is continued. The dominant factor in the coefficient of the x term is $(2a^m r^m \cos m\phi)$. As m continues to grow, this factor may acquire either plus or minus signs, more or less at random, and will in general not reach a point such that it becomes the square of the preceding factor.

The magnitudes of a and r may be evaluated as soon as the doubled product terms in the coefficient of x^2 become negligibly small. When such is the case:

$$a^m = \text{coefficient of the } x^2 \text{ term in the "}m\text{th power equation"}$$

$$a^m r^{2m} = \text{constant term of the "}m\text{th power equation"}$$

Component Parts of r.—The separation of r into its real and imaginary components is an important step in the problem at hand. Physically, the real component represents a damping constant and the imaginary component represents the angular velocity of an electrical oscillation if the equation involved pertains to the current flow in an electric circuit. Performing the operations indicated in (17) the third degree equation takes the following form:

$$x^3 + (2u + a)x^2 + (2ua + r^2)x + ar^2 = 0 \tag{21}$$

The magnitudes of a and r have presumably been evaluated. The coefficient of either the x^2 term or the x term of the original equation may be employed in the evaluation of u. The simpler procedure and the one which is usually the more accurate is to evaluate u from the following relationship:

$$(-2u - a) = \text{coefficient of the } x^2 \text{ term in the original equation}$$

The values of r and u having been determined, the value of v follows directly:

$$v = \sqrt{r^2 - u^2}$$

The value of ϕ may be determined from the values of v and u.

$$\phi = \tan^{-1} \frac{v}{u}$$

due regard being given to the individual signs of v and u.

Particular Solution of a Third Degree Equation. One Pair of Complex Roots

In the mathematical analysis of a series-parallel circuit in Chapter IV a third degree determinantal equation is encountered. The solution of that equation will be given here in detail in order to illustrate further the application of the Graeffe method. For convenience equation (77) of Chapter IV is rewritten at this point.

$$p^3 + 242p^2 + 0.539 \times 10^6 p + 79.8 \times 10^6 = 0 \tag{22}$$

The results of the root-squaring process are given below in tabular form.

The change in sign of the coefficients of the p^2 terms together with the persistence of the doubled products in the p^2 column indicate that a pair of complex roots is present, the modulus of which is larger than the real root. The roots may be evaluated in accordance with the principles previously discussed. In the present case the coefficients of

Term	p^3	p^2	p	Constant
Coefficients of original equation.	1	242	5 39$\times 10^{5}$	79.8$\times 10^{6}$
Squares	1	5 8^{4}	29 0^{10}	63 8^{14}
Doubled products. . .		−107 8^{4}	−3 9^{10}	
Coefficients of 2nd equation. . .	1	−102 0^{4}	25 1^{10}	63 8^{14}
Squares	1	1 04^{12}	6 30^{22}	40 7^{30}
Doubled products..... . . .		−0 50^{12}	1 30^{22}	
Coefficients of 4th equation	1	0 54^{12}	7 60^{22}	40 7^{30}
Squares.................. ...	1	0 29^{24}	5.78^{45}	1.66^{63}
Doubled products		−0.15^{24}	−0.04^{45}	
Coefficients of 8th equation.. ..	1	0 14^{24}	5 74^{45}	1.66^{63}
Squares............	1	1 96^{46}	3 30^{91}	2.76^{126}
Doubled products...		−1.15^{46}	negligible	
Coefficients of 16th equation....	1	0 81^{46}	3.30^{91}	2 76^{126}

the equation whose roots are the 16th power of the original roots may be used to evaluate r and a, hence m is equal to 16.

$$r^{2m} = \text{coefficient of the } p \text{ term in the } m\text{th equation.}$$

$$r^{32} = 3.30 \times 10^{91}$$

$$\log_{10} r = \frac{91.5185}{32}$$

$$\log_{10} r = 2.86$$

$$r = 725, \text{ the modulus of the complex roots}$$

$$a^m r^{2m} = \text{constant term in the } m\text{th equation}$$

$$a^{16} = \frac{2.76 \times 10^{126}}{3.30 \times 10^{91}}$$

$$\log_{10} a = 2.182$$

$$a = 152, \text{ in magnitude}$$

Substitution in the original equation shows that -152 is one of the roots.

$$(2u + a) = -242.$$
$$u = -45.$$
$$v = \sqrt{725^2 - 45^2}$$
$$v = 723.6.$$

The roots of equation (22) are:

(1) $-45 + j723.6$, (2) $-45 - j723.6$, and (3) -152.

Fourth Degree Equations

The solution of a fourth degree equation by the Graeffe method does not differ essentially from that outlined for the third degree equation. Other combinations of roots are possible, of course, when a fourth degree equation is involved, and this necessitates a more detailed analysis of the numerical columns of the "root-squaring" table. The correct interpretation of the type of equation involved and the exact nature of the roots must be made before the correct relationships for the evaluation of the roots can be found.

If the doubled products all vanish as the root-squaring process is carried out and the coefficients in a newly formed equation all become the squares of the corresponding coefficients in the preceding equation, four real roots are indicated. The analyses previously made of the coefficients of equation (14) point to the reasons why such should be the case.

If the coefficients of one of the terms oscillate in sign and the doubled products in this one column fail to vanish as the root-squaring process is continued, two real and two complex roots are indicated. If the coefficients of two of the terms oscillate in sign and the doubled products in these two columns fail to vanish, two pairs of complex roots are indicated. A most satisfactory means of becoming acquainted with the reasons back of the foregoing statements is to solve a fourth degree equation, the roots of which are unknown both as to type and magnitude.

A fourth degree equation, namely, equation (93) of Chapter V, is encountered in the mathematical analysis of magnetically coupled circuits. The solution of that equation will be attempted, assuming that the nature of the roots is unknown. Let it be required to evaluate the roots of the following equation:

$$p^4 + 326p^3 + 1.81\cdot 10^6 p^2 + 7.52\cdot 10^7 p + 1.142\cdot 10^{11} = 0 \quad (23)$$

The results of the root-squaring process are given below in the customary tabular form:

	x^4	x^3	x^2	x	
Term	p^4	p^3	p^2	p	Constant
Coefficients of original equation .	1	326	1.81^{6}	7.52^{7}	$1\ 142^{11}$
Squares.	1	10.6^{4}	3.28^{12}	5.67^{15}	$1\ 31^{22}$
Doubled products.		-362.0^{4}	-0.05^{12} $0\ 23^{12}$	-414.00^{15}	
Coefficients of 2nd equation. . . .	1	-351^{4}	$3\ 46^{12}$	408^{15}	1.31^{22}
Squares.	1	$1\ 24^{13}$	$12\ 0^{24}$	16.6^{34}	1.72^{44}
Doubled products.		$-0\ 69^{13}$	-2.87^{24} $0\ 03^{24}$	-9.08^{34}	
Coefficients of 4th equation. . .	1	$5\ 5^{12}$	$9\ 16^{24}$	$7\ 52^{34}$	$1\ 72^{44}$
Squares.	1	30.3^{24}	$84\ 0^{48}$	$56\ 5^{68}$	$2\ 96^{88}$
Doubled products.		-18.3^{24}	-0.8^{48} $3\ 4^{44}$	-31.5^{68}	
Coefficients of 8th equation. . . .	1	12.0^{24}	83.2^{48}	$25\ 0^{68}$	2.96^{88}
Squares.	1	$144.^{48}$	69.5^{98}	6.25^{138}	8.78^{176}
Doubled products.		$-166.^{48}$	-0.6^{95}	-4.93^{138}	
Coefficients of 16th equation. . . .	1	-22^{48}	69.5^{98}	$1\ 32^{138}$	8.78^{176}

Determining the Nature of the Roots.—The behavior of the coefficients is such as to exclude the possibility of all the roots being real. If our experience in the theory of equations is limited we may admit the possibility of there being two real roots and two complex roots. The error of such an admission soon becomes apparent if a detailed analysis of an equation having two real and two complex roots is made.

An equation of this type might be written as follows:

$$(x + r\epsilon^{j\phi})(x + r\epsilon^{-j\phi})(x + a)(x + b) = 0 \tag{24}$$

In expanded form the above equation becomes:

$$\begin{aligned} x^4 &+ (2r \cos \phi + a + b)x^3 + [2(a + b)r \cos \phi + r^2 + ab]x^2 \\ &+ [2abr \cos \phi + (a + b)r^2]x + abr^2 = 0 \end{aligned} \tag{25}$$

The equation whose roots are the mth power of the roots of the above equation is:

$$x^4+(2r^m\cos m\phi+a^m+b^m)x^3+[2(a^m+b^m)r^m\cos m\phi+r^{2m}+a^mb^m]x^2 + [2a^mb^mr^m\cos m\phi + (a^m + b^m)r^{2m}]x + a^mb^mr^{2m} = 0 \qquad (26)$$

The coefficients of the above equation show that, as the root-squaring process is continued, only one set of coefficients will oscillate in sign and have persistent doubled product terms. If r is greater than a and b the coefficients of x^3 will be the only ones which will oscillate in sign and not lose their doubled product terms. This is due to the fact that $(2r^m \cos m\phi)$ is the dominant factor in the x^3 coefficient when r is greater than a and b. Under these conditions, the coefficients of x^2 are dominated by r^{2m}, and the coefficients of x are dominated by $(a^m + b^m)r^{2m}$.

If r is less than a and b, only the coefficients of x will oscillate in sign and not lose their doubled product terms. Under these conditions the coefficients of x^3 will be dominated by a^m or b^m, and the coefficients of x^2 are dominated by a^mb^m. When m becomes sufficiently large the factors other than the dominant ones may be neglected. The coefficients of x^3 and x^2 will therefore reach a point in the root-squaring process after which a newly formed coefficient is merely the square of the preceding coefficient. $(2a^mb^mr^m \cos m\phi)$ dominates the coefficient of the x term with the result that the coefficients will vary irregularly in magnitude and in sign as the root-squaring process is carried out. The significant point of this analysis is that the coefficients in only one of the columns of the "root-squaring" table will continue to vary irregularly in magnitude and in sign if the equation possesses two real and two complex roots. The fact that the coefficients in two of the columns in the "root-squaring" table vary irregularly in magnitude and sign excludes the possibility of equation (23) having two real and two complex roots.

It remains to examine the coefficients of a fourth degree equation, the roots of which are all complex. In factored form an equation of this type may be written as follows:

$$(x + r_1\epsilon^{j\phi_1})(x + r_1\epsilon^{-j\phi_1})(x + r_2\epsilon^{j\phi_2})(x + r_2\epsilon^{-j\phi_2}) = 0 \qquad (27)$$

In expanded form the equation whose roots are the mth power of the roots of the above equation is:

$$x^4 + (2r_1^m \cos m\phi_1 + 2r_2^m \cos m\phi_2)x^3 + (r_1^{2m} + r_2^{2m} + 4r_1^mr_2^m \cos m\phi_1 \cos m\phi_2)x_2 + (2r_1^{2m}r_2^m \cos m\phi_2 + 2r_2^{2m}r_1^m \cos m\phi_1)x + r_1^{2m}r_2^{2m} = 0 \qquad (28)$$

The general nature of the coefficients of the x^3 term and the x term is such that the newly formed coefficients will not, successively, become the squares of the preceding coefficients as the root-squaring process is carried out. A persistence of the doubled product terms may therefore be expected in the x^3 and the x columns. A random variation in the signs of the coefficients in these columns may also be anticipated.

Since the numerical coefficients in the "root-squaring" table agree in every respect with the anticipated behavior of a fourth degree equation having two pairs of complex roots, the existence of two pairs of complex roots may be assumed. ${r_1}^{2m}$ or ${r_2}^{2m}$ will dominate the coefficients of x^2. If the larger of the two moduli is called r_1, ${r_1}^{2m}$ becomes very much greater than the other factors which go to form the coefficients of x^2. As the root-squaring process is continued the coefficient of x^2 is sensibly equal to ${r_1}^{2m}$, at which point the doubled product terms become negligibly small. The magnitudes of r_1 and r_2 may be evaluated as soon as the doubled product terms in the x^2 column can be neglected.

$${r_1}^{2m} = \text{coefficient of } x^2 \text{ in the } m\text{th power equation}$$

$${r_1}^{2m}{r_2}^{2m} = \text{constant term in the } m\text{th power equation}$$

After r_1 is evaluated the constant term of the original equation may be employed in finding r_2.

$$r_1^2 r_2^2 = \text{constant term of the original equation}$$

Separation of r_1 and r_2 into Their Component Parts.—In order to separate r_1 and r_2 into their component parts further relationships must be established. A detailed examination of the coefficients of a generalized fourth degree equation having four complex roots should provide certain useful relationships.

Let the roots be:

(1) $$-r_1\epsilon^{j\phi_1} = -(u_1 + jv_1)$$

(2) $$-r_1\epsilon^{-j\phi_1} = -(u_1 - jv_1)$$

(3) $$-r_2\epsilon^{j\phi_2} = -(u_2 + jv_2)$$

and

(4) $$-r_2\epsilon^{-j\phi_2} = -(u_2 - jv_2)$$

In terms of u and v, the fourth degree equation may be written as follows:

$$(x + u_1 + jv_1)(x + u_1 - jv_1)(x + u_2 + jv_2)(x + u_2 - jv_2) = 0 \qquad (29)$$

from which:

$$x^4+(2u_1+2u_2)x^3+(4u_1u_2+r_1^2+r_2^2)x^2+(2u_2r_1^2+2u_1r_2^2)x+r_1^2r_2^2=0 \tag{30}$$

Assuming that r_1 and r_2 have been evaluated, three useful relationships between u_1, u_2, r_1, and r_2 have been established. Any two of these relationships may be employed in the determination of u_1 and u_2. From the standpoint of numerical accuracy one relationship may be found to be more desirable than another, in a particular case. The relative values of the u's and r's will determine which of the relationships should be used for greatest accuracy.

$$-(2u_1 + 2u_2) = \text{coefficient of } x^3 \text{ in the original equation}$$

$$-(2u_2r_1^2 + 2u_1r_2^2) = \text{coefficient of } x \text{ in the original equation}$$

The above relationships may be solved simultaneously for u_1 and u_2. The values of the u's having been determined, the v's and ϕ's may be evaluated in the manner previously shown.

Numerical Evaluation of the Roots of Equation (23).—

$$m = 16$$
$$r_1^{2m} = 69.5 \times 10^{98} \qquad r_1^2r_2^2 = 1.142 \times 10^{11}$$
$$r_1^{32} = 69.5 \times 10^{98} \qquad r^2 = \sqrt{65700}$$
$$\log_{10}r_1 = 3.120 \qquad r_2 = 257$$
$$r_1 = 1318$$

$$(2u_1 + 2u_2) = -\ 326$$
$$u_2 = -\ 163 - u_1$$
$$(2u_2r_1^2 + 2u_1r_2^2) = -\ 7.52 \times 10^7$$
$$u_1(r_1^2 - r_2^2) = -\ 24.64 \times 10^7$$
$$u_1 = -\ 146.9$$
$$u_2 = -\ 16.1$$
$$v_1 = 1309$$
$$v_2 = 256$$

The roots of equation (23) which is identical with equation (93) of Chapter V are:

(1) $-146.9 + j1309$ (3) $-16.1 + j256$

(2) $-146.9 - j1309$ (4) $-16.1 - j256$

The above values satisfy the equation to a close degree of accuracy.

The Special Case of Equal Roots

The presence of equal roots is readily detected by the singular behavior of the coefficients in one column of the "root-squaring" table. If the doubled products in a given column fail to vanish and the newly formed coefficient becomes equal to one-half the square of the preceding coefficient, a pair of roots which are equal in magnitude is indicated. This may be illustrated in a particular case by assuming that $r_1 = r_2$ in equation (28). Under these conditions the coefficients of the x^2 terms will behave in a singular manner. Neither ${r_1}^{2m}$ nor ${r_2}^{2m}$ can, by itself, dominate the coefficient of x^2 during the root-squaring process. Hence the newly formed coefficient in the root-squaring process can never acquire a value which is equal to the square of the preceding coefficient. However, $({r_1}^{2m} + {r_2}^{2m})$, which under the condition of equal roots is the same as $2{r_1}^{2m}$, will dominate the coefficient, and as m continues to increase a point will be reached after which the coefficient of x^2 can be set equal to $2{r_1}^{2m}$ without involving an appreciable error. For the sake of further illustration, assume that the coefficient of x^2 is sensibly represented by $2{r_1}^{2m}$ when m is equal to 16.

$$\text{When } m = 16,\ 2{r_1}^{2m} = 2r^{32}$$

$$\text{When } m = 32,\ 2{r_1}^{2m} = 2r^{64}, \text{ etc.}$$

Thus it will be seen that as m increases, a coefficient governed in magnitude by $2{r_1}^{2m}$ will become successively equal to one-half the square of the preceding coefficient. This is the characteristic by which equal roots are recognized in the Graeffe method of solution.

SECTION IV

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

The Number ϵ.—The number ϵ occurs frequently in the mathematical interpretations of natural physical laws and particularly in mathematical expressions which, in some manner, involve *change* or *rate of change* of a physical quantity. ϵ enters into such expressions because the physical phenomenon being described varies exponentially, and exponential ϵ is the most convenient of the exponential forms to manipulate mathematically. Not only does ϵ enter into the solution as a natural consequence; it is sometimes artificially injected in order to simplify the analysis. Modern methods of analysis have been described as "ϵ-methods."

Of and by itself the number ϵ is merely a constant, defined in magnitude by a rapidly converging infinite series. ϵ comes into the mathematical picture during the derivation of the derivative of a logarithm. In the elementary calculus it is shown that:

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a\left[\lim_{\Delta x \to 0}\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right] \tag{1}$$

The bracket term in the above equation takes the form:

$$\left[\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n\right]$$

It is called ϵ. Thus:

$$\epsilon = \left[\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n\right] = 2.71828\ldots \tag{2}$$

The numerical evaluation of the number ϵ is shown in a later paragraph.

Equation (1) now becomes:

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a \epsilon \tag{3}$$

Since a is any arbitrary base, the expression for the derivative of the logarithm is simplified by letting $a = \epsilon$.

If $a = \epsilon$:

$$\frac{d}{dx}\log_\epsilon x = \frac{1}{x} \tag{4}$$

Logarithms to the base ϵ are used almost exclusively in mathematical analyses which involve differentiation and integration. Throughout the text, logarithms to the base ϵ are understood unless otherwise designated.

Exponential Functions

The simplest exponential function with which to deal is ϵ^x.

$$\frac{d}{dx}\epsilon^x = \epsilon^x$$

whereas:

$$\frac{d}{dx}a^x = a^x \log_\epsilon a$$

For any value of x, the rate of change of the function ϵ^x is equal in magnitude to the magnitude of the function itself. Likewise the rate of change of the function ϵ^{-x} is $-\epsilon^{-x}$.

Examples of exponential rates of change are common. The rate at which current decays in the RL circuit is, at any instant, directly proportional to the magnitude of the current in the circuit at that instant. The rate of change in the velocity or the deceleration of a physical body which is controlled by frictional and inertial phenomena is at any instant, directly proportional to the velocity of the body at that instant. The growth of capital at compound interest and even the growth of bacteria under certain conditions are examples of exponential increases. The longer the growth continues the greater will be the rate of growth. Many other physical phenomena may be described in terms of exponential functions. The forms of exponential functions most commonly encountered are ϵ^{ax} and ϵ^{-ax} where a is constant and x is the independent variable.

Expansion in a Maclaurin's Series.—Certain types of functions, of which ϵ^{ax} is one, may be expanded in accordance with Maclaurin's theorem. If the function of x, $f(x)$, is such that it can be expanded in a Maclaurin's series, it takes the following form:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{\underline{|2}}x^2 + \frac{f'''(0)}{\underline{|3}}x^3 + \cdots \tag{5}$$

Obviously, one of the necessary conditions for the existence of the series is that the function and all its derivatives be defined at $x = 0$. Let it be required to expand ϵ^{ax} in a Maclaurin's series. In this case:

$$
\begin{aligned}
f(x) &= \epsilon^{ax} & f(0) &= 1 \\
f'(x) &= a\epsilon^{ax} & f'(0) &= a \\
f''(x) &= a^2\epsilon^{ax} & f''(0) &= a^2 \\
f'''(x) &= a^3\epsilon^{ax} & f'''(0) &= a^3 \\
&\text{etc.} & &\text{etc}
\end{aligned}
$$

$$\epsilon^{ax} = 1 + ax + \frac{a^2x^2}{\underline{|2}} + \frac{a^3x^3}{\underline{|3}} + \cdots \tag{6}$$

$$\epsilon^{-ax} = 1 - ax + \frac{a^2x^2}{\underline{|2}} - \frac{a^3x^3}{\underline{|3}} + \cdots \tag{7}$$

When $ax = 1$:

$$\epsilon^{ax} = \epsilon = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots \tag{8}$$

$$\epsilon = 2.71828 \cdots$$

General Nature of Simple Exponential Functions.—Graphical representations of ϵ^x and ϵ^{-x} are shown in Fig. 1. Combinations of these exponential variations form an important class of functions, namely, hyperbolic functions.

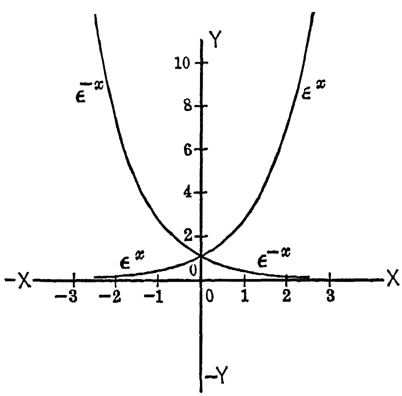


FIG. 1.—Graphs of $y = \epsilon^x$ and of $y = \epsilon^{-x}$.

The expression ϵ^{ax}, or its equivalent, is encountered repeatedly throughout the text. Its derivative and integral should be known and understood.

$$\frac{d}{dx}\epsilon^{ax} = a\epsilon^{ax}$$

$$\int \epsilon^{ax}\,dx = \frac{1}{a}\epsilon^{ax} + c_1$$

Numerical Evaluation.—The numerical values of ϵ^{ax} or ϵ^{-ax} may be obtained in several different ways. For example:

1. Numerical substitution for ax in the expanded forms shown in equations (6) and (7).
2. The use of the log–log slide rule to obtain ϵ^{ax} directly.
3. The use of anti-logarithms as outlined below.

$$
\begin{aligned}
\epsilon^{ax} &= y \\
ax \log_{10} \epsilon &= \log_{10} y \\
y &= \text{anti-}\log_{10}\,[0.4343 \cdot ax]
\end{aligned}
$$

4. Reference to computed values similar to those shown in the table which starts on page 322.

5. Reference to plotted values of ϵ^{ax} and ϵ^{-ax}, similar to those shown in Fig. 2. Plotted on semi-logarithmic paper the functions are represented by straight lines. If numerical values of the functions are to be

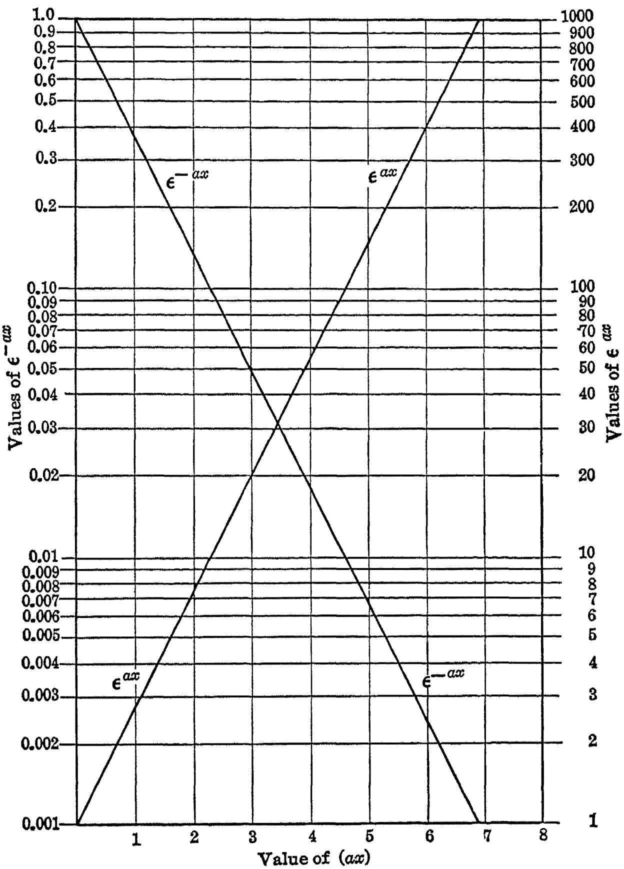


FIG. 2.—Graphs of ϵ^{ax} and ϵ^{-ax} on semi-logarithmic paper.

taken from plotted representations, it is suggested that the plots be made to a larger scale than that of Fig. 2.

ϵ^{jax}, where $j = \sqrt{-1}$.—In addition to having derivatives that are proportional to the function itself, the exponential function possesses another most important property, namely, that

$$\epsilon^{jax} = \cos ax + j \sin ax \tag{9}$$

The above relation is known as Euler's equation and forms the basis of a very powerful method of analysis. A proof of Euler's equation may be established in terms of the expanded forms of ϵ^{jax}, cos ax, and sin ax. Expansion of these functions in accordance with Maclaurin's theorem yields:

1. $\epsilon^{jax} = 1 + jax + \frac{(jax)^2}{\underline{|2}} + \frac{(jax)^3}{\underline{|3}} + \frac{(jax)^4}{\underline{|4}} + \frac{(jax)^5}{\underline{|5}} + \cdots$

$$j = \sqrt{-1},\ j^2 = -1,\ j^3 = -\sqrt{-1},\ j^4 = 1, \text{ etc.}$$

$$\epsilon^{jax} = \left(1 - \frac{a^2x^2}{\underline{|2}} + \frac{a^4x^4}{\underline{|4}} - \frac{a^6x^6}{\underline{|6}} + \text{etc.}\right)$$

$$+ j\left(ax - \frac{a^3x^3}{\underline{|3}} + \frac{a^5x^5}{\underline{|5}} - \text{etc.}\right) \quad (10)$$

2. $\cos ax = 1 - \frac{a^2x^2}{\underline{|2}} + \frac{a^4x^4}{\underline{|4}} - \frac{a^6x^6}{\underline{|6}} + \text{etc.}$ (11)

3. $\sin ax = ax - \frac{a^3x^3}{\underline{|3}} + \frac{a^5x^5}{\underline{|5}} - \text{etc.}$ (12)

An examination of the expanded forms will show that the relation stated in Euler's equation is correct.

In a similar manner it may be shown that:

$$\epsilon^{-jax} = \cos ax - j \sin ax \quad (13)$$

Analytical Forms of Sine and Cosine.—The analytical definition of cos ax and sin ax follow directly from the relations stated in equations (9) and (13). Solving these equations for cos ax and sin ax it may be shown that:

$$\cos ax = \frac{\epsilon^{jax} + \epsilon^{-jax}}{2} \quad (14)$$

and

$$\sin ax = \frac{\epsilon^{jax} - \epsilon^{-jax}}{2j} \quad (15)$$

The Uses of ϵ^{jax} in Circuit Analysis.—The real part of the exponent, ax, represents radian angular measure. In certain cases it is desirable to write ax simply as an angle θ. As such, the operators $\epsilon^{j\theta}$ and $\epsilon^{-j\theta}$ may be used to advantage in connection with vector quantities. $\epsilon^{j\theta}$ is an operator which rotates a vector to which it is applied through an angle

of plus θ degrees. $\epsilon^{-j\theta}$ is an operator which rotates a vector through θ degrees in the negative direction.

The exponential operator may also be employed to produce uniform rotation of a vector. If the angular velocity is to be ω radians per second and time is represented by the symbol t, the operator which will produce uniform rotation of ω radians per second to the vector to which it is applied is:

From the expansions previously given it follows that:

$$\epsilon^{j\omega t} = \cos \omega t + j \sin \omega t \tag{16}$$

A unique method of investigating the behavior of an electric circuit to which alternating potential is applied employs the above relation. It is assumed that the voltage applied to the circuit is:

$$E_m \epsilon^{j\omega t} = E_m(\cos \omega t + j \sin \omega t) \tag{17}$$

The current solutions will therefore consist of two distinct parts, namely: (1) that due to $E_m \cos \omega t$, (2) that due to $jE_m \sin \omega t$.

In a case of this kind the principle of decomposition may be applied. The real part of the solution is " due to " $E_m \cos \omega t$; the j part is " due to " $E_m \sin \omega t$. After separating the solution into its real and j parts it is simply a matter of using one and discarding the other. The manipulation of the exponential function is much more simple and direct than the manipulation of a trigonometric function. For example, the solution of a differential equation whose right-hand member is $E_m \epsilon^{j(\omega t + \lambda)}$ is much more direct than the solution of one whose right-hand member is $E_m \sin (\omega t + \lambda)$. This method although actually very simple is not used in the illustrative examples in the text because it presupposes a certain mathematical maturity on the part of the student which, in all cases, may not exist. The method is strongly recommended to those who are proficient in mathematics because its use will eliminate much of the tedious integration that is encountered in the study of a-c transients.

ϵ^z, where $z = x + jy$.—α_1, α_2, α_3, etc., that appear in the general solutions of the differential equations described in this Appendix[1] are very likely to be complex numbers. Similarly, p_1, p_2, p_3, etc., in Heaviside's expansion formula[2] may be complex in form. These complex numbers are not artificially injected for the sake of simplifying the analysis. They are the roots of the determinantal equation, and as

[1] See Section I.
[2] See Section II.

such they specify and describe the transient response of the circuit which is under investigation. The real parts specify the degree of "damping"; and the j parts, the "natural angular velocities" from which the "natural frequencies" are determined. The mathematical manipulation of ϵ^z, where z is a complex number, is therefore a most important operation in "transient solutions" of electric circuit problems.

One of the outstanding characteristics of the exponential function is the "additive theorem." In elementary algebra it is a formal law that:

$$a^m a^n = a^{m+n} \tag{18}$$

where m and n are real numbers.

The additive theorem applies even though one or both of the terms, m and n, are complex numbers. For example:

$$\begin{aligned} \epsilon^{z_1} \cdot \epsilon^{z_2} &= \epsilon^{x_1} \cdot \epsilon^{jy_1} \cdot \epsilon^{x_2} \cdot \epsilon^{jy_2} \\ &= \epsilon^{(x_1+x_2)} \cdot \epsilon^{j(y_1+y_2)} \end{aligned} \tag{19}$$

Much use is made of this particular fact in the algebraic simplification of transient solutions.

Hyperbolic Functions

Combinations of the simple exponential functions ϵ^{ax} and ϵ^{-ax} occur very frequently. The more frequent combinations are $(\epsilon^{ax} - \epsilon^{-ax})$ and $(\epsilon^{ax} + \epsilon^{-ax})$. As a matter of fact, these combinations are so prevalent in modern mathematical analyses [3] that names have been given to the fractions $(\epsilon^{ax} - \epsilon^{-ax})/2$ and $(\epsilon^{ax} + \epsilon^{-ax})/2$.

By definition:

$$\sinh ax = \frac{\epsilon^{ax} - \epsilon^{-ax}}{2} \tag{20}$$

$$\cosh ax = \frac{\epsilon^{ax} + \epsilon^{-ax}}{2} \tag{21}$$

$$\tanh ax = \frac{\epsilon^{ax} - \epsilon^{-ax}}{\epsilon^{ax} + \epsilon^{-ax}} \tag{22}$$

Sinh ax is read "the hyperbolic sine of ax." The other hyperbolic functions are read correspondingly.

These functions bear relations to the rectangular hyperbola, $x^2 - y^2 = a^2$, which are very similar to those borne by the circular or

[3] The general use of hyperbolic functions is of relatively recent date although the functions have been in existence since 1757.

trigonometric functions to the circle, $x^2 + y^2 = a^2$. However, the analytical treatment of hyperbolic functions is not at all dependent upon the relations borne to the rectangular hyperbola. The detailed geometrical aspect of the functions will not be considered here,[4] inasmuch as it is relatively unimportant in circuit analysis.

Graphical Representation.—Hyperbolic functions may be graphed in terms of ϵ^x and ϵ^{-x}. Fig. 3 illustrates the nature of sinh, cosh, and tanh variations. Note that $\sinh x = 0$ at $x = 0$ and increases positively for positive increases in x. It increases in a similar manner negatively for negative increases in x. Cosh $x = 1$ at $x = 0$ and increases positively for both positive and negative increases in x. Tanh $x = 0$ at $x = 0$ and approaches plus one as a limit for positive increases in x. It likewise approaches negative one as a limit for increasing negative values of x. Sinh x approaches cosh x for large positive values of x.

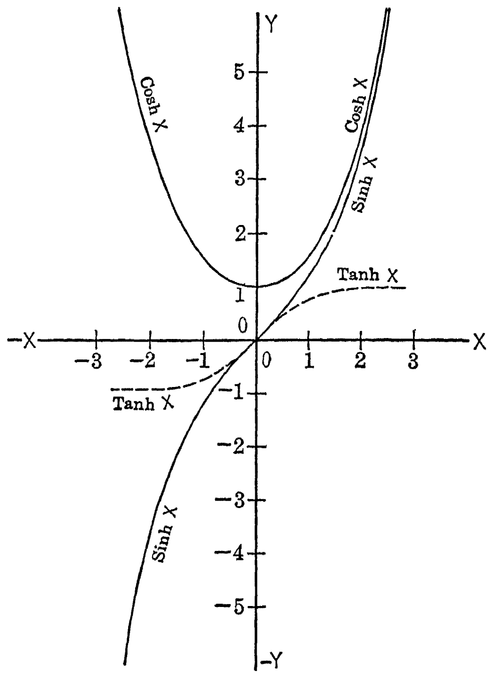


FIG. 3.—Graphs of $y = \sinh x$, $y = \cosh x$, and $y = \tanh x$.

Mathematical Manipulation.—Hyperbolic functions are very simple to manipulate mathematically. Such would be expected of functions composed of ϵ^{ax} and ϵ^{-ax}. The functions and their successive derivatives are defined for all values of the independent variable.

$$\sinh ax = ax + \frac{a^3x^3}{\underline{|3}} + \frac{a^5x^5}{\underline{|5}} + \cdots \tag{23}$$

$$\cosh ax = 1 + \frac{a^2x^2}{\underline{|2}} + \frac{a^4x^4}{\underline{|4}} + \cdots \tag{24}$$

The above series may be established by combining the expanded forms

[4] See Osgood, "Advanced Calculus," page 506.

of ϵ^{ax} and ϵ^{-ax} in accordance with the definitions of sinh and cosh; or the functions themselves may be expanded in Maclaurin's series.

Differentiation and integration of the functions may be carried out in terms of their simple exponential factors if any doubt should arise as to their differentiation or integration. For example:

$$\frac{d}{dx}\cosh ax = \frac{d}{dx}\frac{(\epsilon^{ax} + \epsilon^{-ax})}{2}$$

$$= \frac{a(\epsilon^{ax} - \epsilon^{-ax})}{2}$$

$$= a \sinh ax$$

Numerical Evaluation.—Various means may be employed to find the numerical value of hyperbolic functions. For example:

1. Numerical evaluation in terms of ϵ^{ax} and ϵ^{-ax}.
2. Substitution in the expanded forms of the functions. See equations (23) and (24).
3. Special slide rules.
4. Use of tables. See "Table of Hyperbolic Functions" given at the close of the Appendix.

Inverse Hyperbolic Functions.—The solution of one type of transients problem depends upon the evaluation of inverse hyperbolic functions. Maximum values of the dependent variable are sometimes defined in terms of inverse hyperbolic tangents. Inverse or anti-hyperbolic functions may be obtained from the "Table of Hyperbolic Functions" for the range covered.

The inverse functions are logarithmic in nature. If

$$y = \sinh x$$

then

$$x = \sinh^{-1} y$$

The above is read "x equals the angle whose hyperbolic sine is y." From the definition of the hyperbolic sign:

$$y = \frac{\epsilon^{x} - \epsilon^{-x}}{2}$$

Reducing to quadratic form and solving for ϵ^{x}, the following is obtained:

$$\epsilon^{x} = y \pm \sqrt{y^2 + 1}$$

The lower sign cannot be used because $\epsilon^{x} > 0$ for all real values of x. Therefore:

$$x = \sinh^{-1} y = \log (y + \sqrt{y^2 + 1}) \qquad (25)$$

In a similar manner it may be shown that:

$$\cosh^{-1} y = \log (y \pm \sqrt{y^2 - 1}), \qquad y \geqq 1 \tag{26}$$

and that:

$$\tanh^{-1} y = \frac{1}{2} \log \frac{1 + y}{1 - y}, \qquad -1 < y < 1 \tag{27}$$

Comparison of Circular and Hyperbolic Functions.—The analytical similarity between circular and hyperbolic functions is illustrated in the following table.[5] Many other corresponding relations can be written. Those listed are frequently encountered in electric-circuit analysis and have, for the most part, been proved or established in the foregoing discussion. It will be observed that, from an analytical point of view, the relations listed under "Hyperbolic Functions" are somewhat more simple in character than those listed under "Circular Functions."

FORMAL ANALOGY BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS

	Circular Functions		*Hyperbolic Functions*
1.	$\sin ax = \dfrac{\epsilon^{jax} - \epsilon^{-jax}}{2j}$	1.	$\sinh ax = \dfrac{\epsilon^{ax} - \epsilon^{-ax}}{2}$
2.	$\cos ax = \dfrac{\epsilon^{jax} + \epsilon^{-jax}}{2}$	2.	$\cosh ax = \dfrac{\epsilon^{ax} + \epsilon^{-ax}}{2}$
3.	$\tan ax = \dfrac{\epsilon^{jax} - \epsilon^{-jax}}{j(\epsilon^{jax} + \epsilon^{-jax})}$	3.	$\tanh ax = \dfrac{\epsilon^{ax} - \epsilon^{-ax}}{\epsilon^{ax} + \epsilon^{-ax}}$
4.	$\cot ax = \dfrac{1}{\tan ax}$	4.	$\coth ax = \dfrac{1}{\tanh ax}$
5.	$\sin ax = ax - \dfrac{a^3x^3}{\lfloor 3} + \dfrac{a^5x^5}{\lfloor 5} - \cdots$	5.	$\sinh ax = ax + \dfrac{a^3x^3}{\lfloor 3} + \dfrac{a^5x^5}{\lfloor 5} + \cdots$
6.	$\cos ax = 1 - \dfrac{a^2x^2}{\lfloor 2} + \dfrac{a^4x^4}{\lfloor 4} - \cdots$	6.	$\cosh ax = 1 + \dfrac{a^2x^2}{\lfloor 2} + \dfrac{a^4x^4}{\lfloor 4} + \cdots$
7.	$\sin^2 ax + \cos^2 ax = 1$	7.	$\cosh^2 ax - \sinh^2 ax = 1$
8.	$1 + \tan^2 ax = \sec^2 ax$	8.	$1 - \tanh^2 ax = \operatorname{sech}^2 ax$
9.	$\sin 2ax = 2 \sin ax \cos ax$	9.	$\sinh 2ax = 2 \sinh ax \cosh ax$
10.	$\cos 2ax = \cos^2 ax - \sin^2 ax$	10.	$\cosh 2ax = \cosh^2 ax + \sinh^2 ax$
11.	$\sin (ax \pm b) = \sin ax \cos b \pm \cos ax \sin b$	11.	$\sinh (ax \pm b) = \sinh ax \cosh b \pm \cosh ax \sinh b$
12.	$\cos (ax \pm b) = \cos ax \cos b \mp \sin ax \sin b$	12.	$\cosh (ax \pm b) = \cosh ax \cosh b \pm \sinh ax \sinh b$

[5] The analogy between circular and hyperbolic functions given here is substantially the same as that given in Sokolnikoff's "Higher Mathematics for Engineers and Physicists," pages 221–222.

Circular Functions

13. $\frac{d}{dx}\sin ax = a\cos ax$

14. $\frac{d}{dx}\cos ax = -a\sin ax$

15. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c_1$

16. $\int \cos ax\,dx = \frac{1}{a}\sin ax + c_1$

Hyperbolic Functions

13. $\frac{d}{dx}\sinh ax = a\cosh ax$

14. $\frac{d}{dx}\cosh ax = a\sinh ax$

15. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + c_1$

16. $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + c_1$

TABLE
VALUES AND LOGARITHMS OF EXPONENTIALS AND HYPERBOLIC FUNCTIONS

The following tables give values of e^x, e^{-x}, sinh x, cosh x and tanh x for values of x from 0.00 to 6.00 in intervals of 0.01.

To facilitate computations involving multiplication, the common logarithms of e^x, sinh x, cosh x, and tanh x are also given.

For values of x greater than 6: e^x may be computed from the relationship $e^x = \log^{-1} (x \log_{10} e) = \log^{-1} 0.43429x$; e^{-x} approaches zero, sinh x and cosh x are approximately equal and become 0.5 e^x; and tanh x and coth x have values approximately equal to unity.

Where more accurate values of the exponentials and functions are required they may be computed from the following relationships.

$$e = 2.71828\ 18285 \qquad \frac{1}{e} = 0.36787\ 94412$$

$$M = \log_{10} e = 0.43429\ 44819 \qquad \frac{1}{M} = \log_e 10 = 2.30258\ 50930$$

$$e^x = \log^{-1} Mx \qquad e^{-x} = \log^{-1} - Mx$$

$$\text{Sinh } x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{csch } x = \frac{1}{\sinh x} \qquad \text{sech } x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

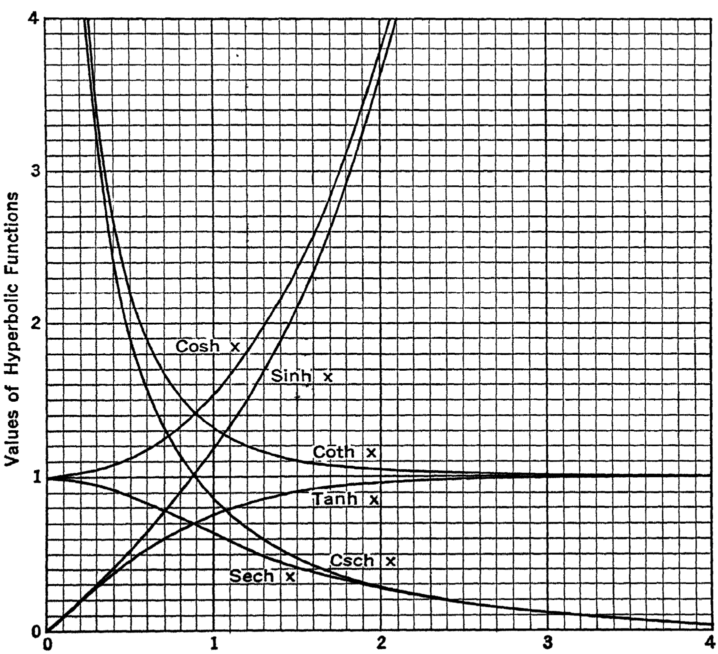


Chart of the Hyperbolic Functions.

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
0.00	**1.0000**	**1.0000**	**0.0000**	**1.0000**	**.00000**	**0.00000**	**$-\infty$**	**0.00000**	**$-\infty$**
0.01	1.0101	.99005	0.0100	1.0001	.01000	.00434	$\bar{2}$.00001	.00002	$\bar{3}$.99999
0.02	1.0202	.98020	0.0200	1.0002	.02000	.00869	.30106	.00009	$\bar{2}$.30097
0.03	1.0305	.97045	0.0300	1.0005	.02999	.01303	.47719	.00020	.47699
0.04	1.0408	.96079	0.0400	1.0008	.03998	.01737	.60218	.00035	.60183
0.05	1.0513	.95123	0.0500	1.0013	.04996	.02171	.69915	.00054	.69861
0.06	1.0618	.94176	0.0600	1.0018	.05993	.02606	.77841	.00078	.77763
0.07	1.0725	.93239	0.0701	1.0025	.06989	.03040	.84545	.00106	.84439
0.08	1.0833	.92312	0.0801	1.0032	.07983	.03474	.90355	.00139	.90216
0.09	1.0942	.91393	0.0901	1.0041	.08976	.03909	.95483	.00176	.95307
0.10	**1.1052**	**.90484**	**0.1002**	**1.0050**	**.09967**	**0.04343**	**$\bar{1}$.00072**	**0.00217**	**$\bar{2}$.99856**
0.11	1.1163	.89583	0.1102	1.0061	.10956	.04777	.04227	.00262	$\bar{1}$.03965
0.12	1.1275	.88692	0.1203	1.0072	.11943	.05212	.08022	.00312	.07710
0.13	1.1388	.87810	0.1304	1.0085	.12927	.05646	.11517	.00366	.11151
0.14	1.1503	.86936	0.1405	1.0098	.13909	.06080	.14755	.00424	.14330
0.15	1.1618	.86071	0.1506	1.0113	.14889	.06514	.17772	.00487	.17285
0.16	1.1735	.85214	0.1607	1.0128	.15865	.06949	.20597	.00554	.20044
0.17	1.1853	.84366	0.1708	1.0145	.16838	.07383	.23254	.00625	.22629
0.18	1.1972	.83527	0.1810	1.0162	.17808	.07817	.25762	.00700	.25062
0.19	1.2092	.82696	0.1911	1.0181	.18775	.08252	.28136	.00779	.27357
0.20	**1.2214**	**.81873**	**0.2013**	**1.0201**	**.19738**	**0.08686**	**$\bar{1}$.30392**	**0.00863**	**$\bar{1}$.29529**
0.21	1.2337	.81058	0.2115	1.0221	.20697	.09120	.32541	.00951	.31590
0.22	1.2461	.80252	0.2218	1.0243	.21652	.09554	.34592	.01043	.33549
0.23	1.2586	.79453	0.2320	1.0266	.22603	.09989	.36555	.01139	.35416
0.24	1.2712	.78663	0.2423	1.0289	.23550	.10423	.38437	.01239	.37198
0.25	1.2840	.77880	0.2526	1.0314	.24492	.10857	.40245	.01343	.38902
0.26	1.2969	.77105	0.2629	1.0340	.25430	.11292	.41986	.01452	.40534
0.27	1.3100	.76338	0.2733	1.0367	.26362	.11726	.43663	.01564	.42099
0.28	1.3231	.75578	0.2837	1.0395	.27291	.12160	.45282	.01681	.43601
0.29	1.3364	.74826	0.2941	1.0423	.28213	.12595	.46847	.01801	.45046
0.30	**1.3499**	**.74082**	**0.3045**	**1.0453**	**.29131**	**0.13029**	**1.48362**	**0.01926**	**$\bar{1}$.46436**
0.31	1.3634	.73345	0.3150	1.0484	.30044	.13463	.49830	.02054	.47775
0.32	1.3771	.72615	0.3255	1.0516	.30951	.13897	.51254	.02107	.49067
0.33	1.3910	.71892	0.3360	1.0549	.31852	.14332	.52637	.02323	.50314
0.34	1.4049	.71177	0.3466	1.0584	.32748	.14766	.53981	.02463	.51518
0.35	1.4191	.70469	0.3572	1.0619	.33638	.15200	.55290	.02607	.52682
0.36	1.4333	.69768	0.3678	1.0655	.34521	.15635	.56564	.02755	.53809
0.37	1.4477	.69073	0.3785	1.0692	.35399	.16069	.57807	.02907	.54899
0.38	1.4623	.68386	0.3892	1.0731	.36271	.16503	.59019	.03063	.55956
0.39	1.4770	.67706	0.4000	1.0770	.37136	.16937	.60202	.03222	.56980
0.40	**1.4918**	**.67032**	**0.4108**	**1.0811**	**.37995**	**0.17372**	**$\bar{1}$.61358**	**0.03385**	**$\bar{1}$.57973**
0.41	1.5068	.66365	0.4216	1.0852	.38847	.17806	.62488	.03552	.58936
0.42	1.5220	.65705	0.4325	1.0895	.39693	.18240	.63594	.03723	.59871
0.43	1.5373	.65051	0.4434	1.0939	.40532	.18675	.64677	.03897	.60780
0.44	1.5527	.64404	0.4543	1.0984	.41364	.19109	.65738	.04075	.61663
0.45	1.5683	.63763	0.4653	1.1030	.42190	.19543	.66777	.04256	.62521
0.46	1.5841	.63128	0.4764	1.1077	.43008	.19978	.67797	.04441	.63355
0.47	1.6000	.62500	0.4875	1.1125	.43820	.20412	.68797	.04630	.64167
0.48	1.6161	.61878	0.4986	1.1174	.44624	.20846	.69779	.04822	.64957
0.49	1.6323	.61263	0.5098	1.1225	.45422	.21280	.70744	.05018	.65726
0.50	**1.6487**	**.60653**	**0.5211**	**1.1276**	**.46212**	**0.21715**	**1.71692**	**0.05217**	**$\bar{1}$.66475**
0.51	1.6653	.60050	0.5324	1.1329	.46995	.22149	.72624	.05419	.67205
0.52	1.6820	.59452	0.5438	1.1383	.47770	.22583	.73540	.05625	.67916
0.53	1.6989	.58860	0.5552	1.1438	.48538	.23018	.74442	.05834	.68608
0.54	1.7160	.58275	0.5666	1.1494	.49299	.23452	.75330	.06046	.69284
0.55	1.7333	.57695	0.5782	1.1551	.50052	.23886	.76204	.06262	.69942
0.56	1.7507	.57121	0.5897	1.1609	.50798	.24320	.77065	.06481	.70584
0.57	1.7683	.56553	0.6014	1.1669	.51536	.24755	.77914	.06703	.71211
0.58	1.7860	.55990	0.6131	1.1730	.52267	.25189	.78751	.06929	.71822
0.59	1.8040	.55433	0.6248	1.1792	.52990	.25623	.79576	.07157	.72419
0.60	**1.8221**	**.54881**	**0.6367**	**1.1855**	**.53705**	**0.26058**	**$\bar{1}$.80390**	**0.07389**	**$\bar{1}$.73001**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
0.60	**1.8221**	**.54881**	**0.6367**	**1.1855**	**.53705**	**0.26058**	**$\bar{1}$.80390**	**0.07389**	**$\bar{1}$.73001**
0.61	1.8404	.54335	0.6485	1.1919	.54413	.26492	.81194	.07624	.73570
0.62	1.8589	.53794	0.6605	1.1984	.55113	.26926	.81987	07861	.74125
0.63	1.8776	.53259	0.6725	1.2051	.55805	.27361	.82770	.08102	.74667
0.64	1.8965	.52729	0.6846	1.2119	.56490	.27795	.83543	.08346	.75197
0.65	1.9155	.52205	0.6967	1.2188	.57167	.28229	.84308	08593	.75715
0.66	1.9348	.51685	0.7090	1.2258	.57836	.28663	.85063	.08843	.76220
0.67	1.9542	.51171	0.7213	1.2330	.58498	.29098	.85809	.09095	.76714
0.68	1.9739	.50662	0.7336	1.2402	.59152	.29532	.86548	.09351	.77197
0.69	1.9937	.50158	0.7461	1.2476	.59798	.29966	.87278	.09609	.77669
0.70	**2.0138**	**.49659**	**0.7586**	**1.2552**	**.60437**	**0.30401**	**$\bar{1}$.88000**	**0.09870**	**$\bar{1}$.78130**
0.71	2.0340	.49164	0.7712	1.2628	.61068	.30835	.88715	.10134	.78581
0.72	2.0544	.48675	0.7838	1.2706	.61691	.31269	.89423	.10401	.79022
0.73	2.0751	.48191	0.7966	1.2785	.62307	31704	.90123	.10670	.79453
0.74	2.0959	.47711	0.8094	1.2865	.62915	.32138	.90817	.10942	.79875
0.75	2.1170	.47237	0.8223	1.2947	.63515	.32572	.91504	11216	.80288
0.76	2.1383	.46767	0.8353	1.3030	.64108	.33006	.92185	.11493	.80691
0.77	2.1598	.46301	0.8484	1.3114	.64693	.33441	.92859	.11773	.81086
0.78	2.1815	.45841	0.8615	1.3199	.65271	.33875	.93527	.12055	.81472
0.79	2.2034	.45384	0.8748	1.3286	.65841	.34309	.94190	.12340	.81850
0.80	**2.2255**	**.44933**	**0.8881**	**1.3374**	**.66404**	**0.34744**	**$\bar{1}$.94846**	**0.12627**	**$\bar{1}$.82219**
0.81	2.2479	.44486	0.9015	1.3464	.66959	.35178	.95498	.12917	.82581
0.82	2.2705	.44043	0.9150	1.3555	.67507	.35612	.96144	.13209	.82935
0.83	2.2933	.43605	0.9286	1.3647	.68048	.36046	.96784	.13503	.83281
0.84	2.3164	.43171	0.9423	1.3740	.68581	.36481	.97420	.13800	.83620
0.85	2.3396	.42741	0.9561	1.3835	.69107	.36915	.98051	.14099	.83952
0.86	2.3632	.42316	0.9700	1.3932	.69626	37349	.98677	.14400	.84277
0.87	2.3869	.41895	0.9840	1.4029	.70137	.37784	.99299	.14704	.84595
0.88	2.4109	.41478	0.9981	1.4128	.70642	.38218	.99916	.15009	.84906
0.89	2.4351	.41066	1.0122	1.4229	.71139	.38652	0.00528	.15317	.85211
0.90	**2.4596**	**.40657**	**1.0265**	**1.4331**	**.71630**	**0.39087**	**0.01137**	**0.15627**	**$\bar{1}$.85509**
0.91	2.4843	.40252	1.0409	1.4434	.72113	.39521	.01741	.15939	.85801
0.92	2.5093	.39852	1.0554	1.4539	.72590	.39955	.02341	.16254	.86088
0.93	2.5345	.39455	1.0700	1.4645	.73059	.40389	.02937	.16570	.86368
0.94	2.5600	.39063	1.0847	1.4753	.73522	.40824	03530	.16888	.86642
0.95	2.5857	.38674	1.0995	1.4862	.73978	.41258	04119	.17208	.86910
0.96	2.6117	.38289	1.1144	1.4973	.74428	.41692	.04704	.17531	.87173
0.97	2.6379	.37908	1.1294	1.5085	.74870	.42127	.05286	.17855	.87431
0.98	2.6645	.37531	1.1446	1.5199	.75307	.42561	.05864	.18181	.87683
0.99	2.6912	.37158	1.1598	1.5314	.75736	.42995	.06439	.18509	.87930
1.00	**2.7183**	**.36788**	**1.1752**	**1.5431**	**.76159**	**0.43429**	**0.07011**	**0.18839**	**$\bar{1}$.88172**
1.01	2.7456	.36422	1.1907	1.5549	.76576	.43864	07580	.19171	.88409
1.02	2.7732	.36059	1.2063	1.5669	.76987	.44298	.08146	.19504	.88642
1.03	2.8011	.35701	1.2220	1.5790	.77391	.44732	.08708	.19839	.88869
1.04	2.8292	.35345	1.2379	1.5913	77789	.45167	.09268	.20176	.89092
1.05	2.8577	.34994	1.2539	1.6038	.78181	.45601	.09825	.20515	.89310
1.06	2.8864	.34646	1.2700	1.6164	.78566	.46035	10379	.20855	.89524
1.07	2.9154	.34301	1.2862	1.6292	.78946	.46470	.10930	.21197	.89733
1.08	2.9447	.33960	1.3025	1.6421	.79320	.46904	.11479	.21541	.89938
1.09	2.9743	.33622	1.3190	1.6552	.79688	.47338	.12025	.21886	.90139
1.10	**3.0042**	**.33287**	**1.3356**	**1.6685**	**.80050**	**0.47772**	**0.12569**	**0.22233**	**$\bar{1}$.90336**
1.11	3.0344	.32956	1.3524	1.6820	.80406	.48207	.13111	.22582	.90529
1.12	3.0649	.32628	1.3693	1.6956	.80757	.48641	.13649	.22931	.90718
1.13	3.0957	.32303	1.3863	1.7093	.81102	.49075	.14186	.23283	.90903
1.14	3.1268	.31982	1.4035	1.7233	.81441	.49510	.14720	.23636	.91085
1.15	3.1582	.31664	1.4208	1.7374	.81775	.49944	.15253	.23990	.91262
1.16	3.1899	.31349	1.4382	1.7517	.82104	.50378	.15783	.24346	.91436
1.17	3.2220	.31037	1.4558	1.7662	.82427	.50812	.16311	.24703	.91607
1.18	3.2544	.30728	1.4735	1.7808	.82745	.51247	.16836	.25062	.91774
1.19	3.2871	.30422	1.4914	1.7957	.83058	.51681	.17360	.25422	.91938
1.20	**3.3201**	**.30119**	**1.5095**	**1.8107**	**.83365**	**0.52115**	**0.17882**	**0.25784**	**$\bar{1}$.92099**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
1 20	**3 3201**	**.30119**	**1.5095**	**1.8107**	**.83365**	**0.52115**	**0.17882**	**0.25784**	**$\bar{1}$ 92099**
1.21	3.3535	.29820	1.5276	1.8258	.83668	.52550	.18402	.26146	.92256
1.22	3 3872	.29523	1 5460	1.8412	.83965	.52984	.18920	.26510	.92410
1 23	3.4212	.29229	1.5645	1.8568	.84258	.53418	.19437	.26876	.92561
1 24	3.4556	.28938	1 5831	1.8725	.84546	.53853	.19951	.27242	.92709
1 25	3 4903	.28650	1.6019	1.8884	.84828	.54287	20464	.27610	.92854
1 26	3.5254	.28365	1.6209	1.9045	.85106	.54721	.20975	.27979	.92996
1.27	3 5609	.28083	1 6400	1.9208	.85380	.55155	.21485	.28349	.93135
1 28	3 5966	.27804	1.6593	1.9373	.85648	.55590	.21993	.28721	.93272
1 29	3.6328	.27527	1.6788	1.9540	.85913	.56024	.22499	.29093	.93406
1 30	**3.6693**	**.27253**	**1.6984**	**1.9709**	**.86172**	**0.56458**	**0.23004**	**0.29467**	**$\bar{1}$.93537**
1.31	3 7062	.26982	1 7182	1.9880	.86428	.56893	.23507	.29842	.93665
1 32	3 7434	.26714	1.7381	2 0053	86678	.57327	.24009	.30217	.93791
1.33	3.7810	.26448	1.7583	2.0228	.86925	.57761	.24509	.30594	.93914
1.34	3 8190	26185	1 7786	2.0404	.87167	.58195	.25008	.30972	.94035
1 35	3 8574	.25924	1.7991	2.0583	.87405	.58630	.25505	.31352	.94154
1 36	3.8962	.25666	1.8198	2 0764	.87639	.59064	.26002	.31732	.94270
1.37	3 9354	25411	1.8406	2 0947	.87869	.59498	.26496	.32113	.94384
1 38	3.9749	.25158	1.8617	2 1132	.88095	.59933	.26990	.32495	.94495
1.39	4.0149	.24908	1.8829	2.1320	.88317	.60367	.27482	.32878	.94604
1.40	**4.0552**	**.24660**	**1.9043**	**2.1509**	**.88535**	**0.60801**	**0.27974**	**0.33262**	**$\bar{1}$.94712**
1.41	4.0960	.24414	1.9259	2.1700	.88749	.61236	.28464	.33647	.94817
1.42	4.1371	.24171	1.9477	2.1894	.88960	.61670	.28952	.34033	.94919
1.43	4 1787	.23931	1 9697	2.2090	.89167	.62104	.29440	.34420	.95020
1.44	4 2207	.23693	1 9919	2.2288	.89370	.62538	.29926	.34807	.95119
1 45	4 2631	.23457	2 0143	2.2488	.89569	.62973	.30412	.35196	.95216
1.46	4.3060	.23224	2 0369	2.2691	.89765	.63407	.30896	.35585	.95311
1.47	4.3492	.22993	2 0597	2.2896	.89958	.63841	.31379	.35976	.95404
1.48	4.3929	22764	2 0827	2 3103	.90147	.64276	.31862	.36367	.95495
1 49	4 4371	.22537	2 1059	2.3312	.90332	.64710	.32343	.36759	.95584
1 50	**4 4817**	**.22313**	**2.1293**	**2.3524**	**.90515**	**0.65144**	**0.32823**	**0.37151**	**$\bar{1}$.95672**
1.51	4.5267	.22091	2.1529	2 3738	.90694	.65578	.33303	.37545	.95758
1.52	4.5722	.21871	2.1768	2.3955	.90870	.66013	.33781	.37939	.95842
1.53	4.6182	.21654	2.2008	2.4174	.91042	.66447	.34258	.38334	.95924
1 54	4 6646	.21438	2.2251	2.4395	.91212	.66881	.34735	.38730	.96005
1.55	4.7115	.21225	2.2496	2.4619	.91379	.67316	.35211	.39126	.96084
1 56	4.7588	.21014	2.2743	2.4845	.91542	.67750	.35686	.39524	.96162
1 57	4.8066	.20805	2.2993	2.5073	.91703	.68184	.36160	.39921	.96238
1 58	4 8550	.20598	2.3245	2.5305	.91860	.68619	.36633	.40320	.96313
1.59	4 9037	.20393	2 3499	2.5538	.92015	.69053	.37105	.40719	.96386
1.60	**4 9530**	**.20190**	**2.3756**	**2.5775**	**.92167**	**0.69487**	**0.37577**	**0.41119**	**$\bar{1}$.96457**
1 61	5 0028	.19989	2.4015	2.6013	.92316	.69921	.38048	.41520	.96528
1 62	5.0531	.19790	2.4276	2.6255	.92462	.70356	.38518	.41921	.96597
1 63	5.1039	.19593	2.4540	2.6499	.92606	.70790	.38987	.42323	.96664
1.64	5 1552	.19398	2 4806	2.6746	.92747	.71224	.39456	.42725	.96730
1.65	5 2070	.19205	2.5075	2.6995	.92886	.71659	.39923	.43129	.96795
1.66	5 2593	.19014	2.5346	2.7247	.93022	.72093	.40391	.43532	.96858
1.67	5.3122	.18825	2.5620	2.7502	.93155	.72527	.40857	.43937	.96921
1 68	5 3656	.18637	2.5896	2.7760	.93286	.72961	.41323	.44341	.96982
1.69	5.4195	.18452	2.6175	2.8020	.93415	.73396	.41788	.44747	.97042
1.70	**5.4739**	**.18268**	**2.6456**	**2.8283**	**.93541**	**0.73830**	**0.42253**	**0.45153**	**$\bar{1}$.97100**
1.71	5.5290	.18087	2.6740	2 8549	.93665	.74264	.42717	.45559	.97158
1.72	5 5845	.17907	2.7027	2.8818	.93786	.74699	.43180	.45966	.97214
1 73	5.6407	.17728	2.7317	2.9090	.93906	.75133	.43643	.46374	.97269
1 74	5 6973	.17552	2.7609	2.9364	.94023	.75567	.44105	.46782	.97323
1 75	5.7546	.17377	2.7904	2.9642	.94138	.76002	.44567	.47191	.97376
1 76	5 8124	.17204	2.8202	2.9922	.94250	.76436	.45028	.47600	.97428
1.77	5 8709	.17033	2 8503	3 0206	.94361	.76870	.45488	.48009	.97479
1 78	5.9299	.16864	2.8806	3.0492	.94470	.77304	.45948	.48419	.97529
1 79	5 9895	.16696	2 9112	3 0782	.94576	.77739	.46408	.48830	.97578
1.80	**6.0496**	**.16530**	**2 9422**	**3.1075**	**.94681**	**0.78173**	**0.46867**	**0.49241**	**$\bar{1}$.97626**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
1.80	**6.0496**	**.16530**	**2.9422**	**3 1075**	**.94681**	**0.78173**	**0.46867**	**0.49241**	**$\bar{1}$.97626**
1 81	6.1104	.16365	2.9734	3 1371	.94783	.78607	.47325	.49652	.97673
1.82	6.1719	.16203	3.0049	3 1669	.94884	.79042	.47783	.50064	.97719
1.83	6.2339	.16041	3.0367	3.1972	.94983	.79476	.48241	.50476	.97764
1.84	6.2965	.15882	3.0689	3.2277	.95080	.79910	.48698	.50889	.97809
1.85	6 3598	.15724	3.1013	3 2585	.95175	.80344	.49154	.51302	.97852
1.86	6.4237	.15567	3.1340	3.2897	.95268	.80779	.49610	.51716	.97895
1.87	6 4883	.15412	3.1671	3.3212	.95359	.81213	.50066	52130	.97936
1.88	6 5535	.15259	3.2005	3 3530	.95449	.81647	.50521	.52544	.97977
1.89	6.6194	.15107	3.2341	3.3852	.95537	.82082	.50976	.52959	.98017
1.90	**6.6859**	**.14957**	**3.2682**	**3.4177**	**.95624**	**0.82516**	**0.51430**	**0 53374**	**$\bar{1}$.98057**
1.91	6.7531	.14808	3.3025	3.4506	.95709	.82950	.51884	.53789	.98095
1.92	6 8210	.14661	3 3372	3 4838	.95792	.83385	.52338	.54205	.98133
1.93	6 8895	.14515	3.3722	3.5173	.95873	.83819	.52791	.54621	.98170
1.94	6 9588	.14370	3 4075	3.5512	.95953	.84253	.53244	.55038	.98206
1.95	7 0287	.14227	3.4432	3.5855	.96032	.84687	.53696	.55455	.98242
1.96	7.0993	.14086	3.4792	3.6201	.96109	.85122	.54148	.55872	.98272
1.97	7 1707	.13946	3.5156	3.6551	96185	.85556	.54600	.56290	.98311
1.98	7 2427	.13807	3 5523	3.6904	.96259	.85990	.55051	.56707	.98344
1.99	7.3155	.13670	3.5894	3.7261	.96331	.86425	.55502	.57126	.98377
2.00	**7.3891**	**.13534**	**3 6269**	**3.7622**	**.96403**	**0.86859**	**0 55953**	**0.57544**	**$\bar{1}$.98409**
2.01	7.4633	.13399	3 6647	3.7987	.96473	.87293	.56403	.57963	.98440
2.02	7.5383	.13266	3.7028	3.8355	.96541	.87727	.56853	.58382	.98471
2.03	7.6141	.13134	3.7414	3.8727	.96609	.88162	.57303	.58802	.98502
2.04	7 6906	.13003	3 7803	3.9103	.96675	.88596	.57753	.59221	.98531
2.05	7.7679	.12873	3 8196	3.9483	.96740	.89030	.58202	.59641	.98560
2.06	7.8460	.12745	3.8593	3.9867	.96803	.89465	.58650	.60061	.98589
2.07	7.9248	.12619	3 8993	4.0255	.96865	.89899	.59099	.60482	.98617
2.08	8 0045	.12493	3.9398	4.0647	.96926	.90333	.59547	.60903	.98644
2.09	8.0849	.12369	3.9806	4.1043	.96986	.90768	.59995	.61324	.98671
2.10	**8.1662**	**.12246**	**4.0219**	**4.1443**	**.97045**	**0.91202**	**0.60443**	**0.61745**	**$\bar{1}$.98697**
2.11	8 2482	.12124	4.0635	4.1847	.97103	.91636	.60890	.62167	.98723
2 12	8 3311	.12003	4.1056	4.2256	.97159	.92070	.61337	.62589	.98748
2.13	8.4149	.11884	4 1480	4.2669	.97215	.92505	.61784	.63011	.98773
2.14	8.4994	.11765	4.1909	4.3085	.97269	.92939	.62231	.63433	.98798
2.15	8.5849	.11648	4.2342	4.3507	.97323	.93373	.62677	.63856	.98821
2.16	8 6711	.11533	4.2779	4.3932	.97375	.93808	.63123	.64278	.98845
2.17	8.7583	.11418	4.3221	4.4362	.97426	.94242	.63569	.64701	.98868
2.18	8 8463	.11304	4.3666	4.4797	.97477	.94676	.64015	.65125	.98890
2.19	8.9352	.11192	4.4116	4.5236	.97526	.95110	.64460	.65548	.98912
2.20	**9.0250**	**.11080**	**4.4571**	**4.5679**	**.97574**	**0 95545**	**0.64905**	**0.65972**	**$\bar{1}$.98934**
2.21	9.1157	.10970	4.5030	4.6127	.97622	95979	.65350	.66396	.98955
2 22	9 2073	.10861	4 5494	4 6580	.97668	.96413	.65795	.66820	98975
2.23	9.2999	.10753	4.5962	4.7037	.97714	.96848	.66240	.67244	.98996
2.24	9.3933	.10646	4.6434	4.7499	.97759	.97282	.66684	.67668	.99016
2.25	9.4877	.10540	4.6912	4.7966	.97803	.97716	.67128	.68093	.99035
2 26	9.5831	.10435	4.7394	4.8437	.97846	.98151	.67572	.68518	.99054
2.27	9.6794	.10331	4.7880	4.8914	.97888	.98585	.68016	.68943	.99073
2.28	9.7767	.10228	4.8372	4.9395	.97929	.99019	.68459	69368	.99091
2.29	9.8749	.10127	4.8868	4.9881	.97970	.99453	.68903	.69794	.99109
2.30	**9.9742**	**.10026**	**4.9370**	**5.0372**	**.98010**	**0.99888**	**0.69346**	**0.70219**	**$\bar{1}$.99127**
2.31	10.074	.09926	4.9876	5.0868	.98049	1 00322	.69789	.70645	.99144
2.32	10.176	.09827	5 0387	5 1370	.98087	.00756	.70232	.71071	.99161
2.33	10.278	.09730	5 0903	5 1876	.98124	.01191	.70675	.71497	.99178
2.34	10 381	.09633	5 1425	5.2388	.98161	01625	.71117	.71923	.99194
2.35	10.486	.09537	5 1951	5 2905	.98197	.02059	.71559	.72349	.99210
2.36	10 591	.09442	5 2483	5.3427	.98233	.02493	.72002	72776	.99226
2.37	10.697	.09348	5.3020	5 3954	.98267	.02928	.72444	.73203	.99241
2.38	10.805	.09255	5 3562	5 4487	.98301	.03362	.72885	.73630	.99256
2.39	10.913	.09163	5.4109	5.5026	.98335	.03796	.73327	.74056	.99271
2.40	**11.023**	**.09072**	**5.4662**	**5.5569**	**98367**	**1.04231**	**0.73769**	**0.74484**	**$\bar{1}$.99285**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
2.40	**11 023**	**.09072**	**5.4662**	**5.5569**	**.98367**	**1.04231**	**0 73769**	**0.74484**	**$\bar{1}$.99285**
2.41	11.134	.08982	5.5221	5 6119	.98400	.04665	.74210	.74911	.99299
2.42	11.246	.08892	5.5785	5.6674	.98431	.05099	.74652	.75338	.99313
2.43	11.359	.08804	5.6354	5 7235	.98462	.05534	.75093	.75766	.99327
2.44	11 473	.08716	5.6929	5 7801	.98492	.05968	.75534	76194	99340
2 45	11.588	.08629	5.7510	5 8373	.98522	.06402	.75975	.76621	99353
2.46	11.705	.08543	5.8097	5.8951	.98551	.06836	.76415	.77049	.99366
2.47	11 822	.08458	5 8689	5.9535	.98579	.07271	.76856	.77477	99379
2.48	11 941	.08374	5 9288	6 0125	.98607	.07705	.77296	.77906	99391
2.49	12.061	.08291	5.9892	6.0721	.98635	.08139	.77737	78334	.99403
2.50	**12.182**	**.08208**	**6.0502**	**6.1323**	**.98661**	**1 08574**	**0.78177**	**0 78762**	**$\bar{1}$ 99415**
2.51	12.305	.08127	6.1118	6.1931	.98688	09008	.78617	.79191	.99426
2.52	12 429	.08046	6.1741	6.2545	.98714	.09442	.79057	.79619	.99438
2.53	12 554	.07966	6.2369	6 3166	.98739	.09877	.79497	.80048	99449
2.54	12 680	07887	6.3004	6 3793	.98764	.10311	.79937	.80477	.99460
2 55	12 807	07808	6 3645	6.4426	.98788	.10745	.80377	.80906	.99470
2.56	12.936	07730	6.4293	6.5066	.98812	.11179	.80816	.81335	.99481
2.57	13 066	.07654	6 4946	6.5712	.98835	.11614	.81256	.81764	.99491
2.58	13 197	.07577	6 5607	6 6365	.98858	.12048	.81695	.82194	.99501
2.59	13 330	.07502	6.6274	6.7024	.98881	.12482	.82134	.82623	.99511
2.60	**13 464**	**.07427**	**6.6947**	**6.7690**	**.98903**	**1.12917**	**0.82573**	**0 83052**	**$\bar{1}$.99521**
2.61	13.599	.07353	6 7628	6 8363	.98924	.13351	.83012	.83482	.99530
2 62	13.736	.07280	6.8315	6.9043	.98946	.13785	.83451	.83912	.99540
2.63	13.874	.07208	6.9008	6 9729	.98966	.14219	.83890	.84341	.99549
2.64	14 013	.07136	6.9709	7.0423	.98987	.14654	.84329	.84771	99558
2.65	14.154	.07065	7.0417	7.1123	.99007	.15088	.84768	.85201	.99566
2.66	14.296	.06995	7.1132	7.1831	.99026	.15522	.85206	.85631	.99575
2.67	14 440	.06925	7.1854	7 2546	.99045	.15957	.85645	.86061	.99583
2.68	14.585	.06856	7.2583	7 3268	.99064	.16391	.86083	.86492	.99592
2.69	14.732	.06788	7.3319	7.3998	.99083	.16825	.86522	.86922	99600
2.70	**14.880**	**.06721**	**7.4063**	**7.4735**	**.99101**	**1.17260**	**0.86960**	**0.87352**	**$\bar{1}$ 99608**
2.71	15.029	.06654	7.4814	7.5479	.99118	.17694	.87398	.87783	.99615
2 72	15.180	.06587	7.5572	7 6231	.99136	.18128	.87836	.88213	.99623
2.73	15.333	.06522	7.6338	7.6991	.99153	.18562	.88274	.88644	.99631
2.74	15.487	.06457	7 7112	7.7758	.99170	.18997	.88712	.89074	.99638
2 75	15.643	.06393	7.7894	7.8533	.99186	.19431	.89150	.89505	.99645
2.76	15.800	.06329	7.8683	7.9316	.99202	.19865	.89588	.89936	.99652
2.77	15.959	.06266	7.9480	8 0106	.99218	.20300	.90026	.90367	.99659
2.78	16.119	.06204	8.0285	8.0905	.99233	.20734	.90463	90798	.99666
2.79	16.281	.06142	8.1098	8.1712	.99248	.21168	.90901	.91229	.99672
2.80	**16.445**	**.06081**	**8.1919**	**8.2527**	**.99263**	**1 21602**	**0.91339**	**0.91660**	**$\bar{1}$.99679**
2.81	16.610	.06020	8.2749	8.3351	.99278	.22037	.91776	.92091	.99685
2.82	16.777	.05961	8 3586	8.4182	.99292	.22471	92213	.92522	.99691
2.83	16.945	.05901	8.4432	8.5022	.99306	.22905	.92651	.92953	.99698
2.84	17.116	.05843	8.5287	8.5871	.99320	.23340	.93088	.93385	.99704
2.85	17.288	.05784	8 6150	8.6728	.99333	23774	.93525	.93816	.99709
2.86	17.462	05727	8.7021	8.7594	.99346	.24208	.93963	.94247	.99715
2.87	17.637	.05670	8.7902	8.8469	.99359	.24643	.94400	.94679	.99721
2 88	17.814	.05613	8 8791	8.9352	.99372	.25077	.94837	.95110	.99726
2.89	17.993	.05558	8 9689	9 0244	.99384	.25511	.95274	.95542	.99732
2.90	**18.174**	**.05502**	**9.0596**	**9.1146**	**.99396**	**1.25945**	**0.95711**	**0.95974**	**$\bar{1}$.99737**
2 91	18.357	.05448	9 1512	9 2056	.99408	.26380	.96148	.96405	.99742
2.92	18 541	.05393	9 2437	9.2976	.99420	.26814	.96584	.96837	.99747
2.93	18.728	.05340	9 3371	9.3905	.99431	.27248	.97021	.97269	.99752
2 94	18.916	.05287	9 4315	9.4844	.99443	.27683	.97458	.97701	.99757
2.95	19.106	.05234	9.5268	9 5791	.99454	.28117	.97895	.98133	.99762
2.96	19 298	.05182	9.6231	9.6749	.99464	.28551	.98331	.98565	.99767
2.97	19 492	.05130	9 7203	9.7716	.99475	.28985	.98768	.98997	.99771
2.98	19.688	.05079	9.8185	9 8693	.99485	.29420	.99205	.99429	.99776
2.99	19 886	.05029	9.9177	9.9680	.99496	.29854	.99641	.99861	.99780
3.00	**20.086**	**.04979**	**10.018**	**10.068**	**.99505**	**1.30288**	**1.00078**	**1.00293**	**$\bar{1}$.99785**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
3.00	**20.086**	**.04979**	**10.018**	**10.068**	**.99505**	**1.30288**	**1.00078**	**1.00293**	**$\bar{1}$.99785**
3.01	20.287	.04929	10.119	10.168	.99515	.30723	.00514	.00725	.99789
3.02	20.491	.04880	10.221	10.270	.99525	.31157	.00950	.01157	.99793
3.03	20.697	.04832	10.325	10.373	.99534	.31591	.01387	.01589	.99797
3.04	20.905	.04783	10.429	10.477	.99543	.32026	.01823	.02022	.99801
3.05	21.115	.04736	10.534	10.581	.99552	.32460	.02259	.02454	.99805
3.06	21.328	.04689	10.640	10.687	.99561	.32894	.02696	.02886	.99809
3.07	21.542	.04642	10.748	10.794	.99570	.33328	.03132	.03319	.99813
3.08	21.758	.04596	10.856	10.902	.99578	.33763	.03568	.03751	.99817
3.09	21.977	.04550	10.966	11.011	.99587	.34197	.04004	.04184	.99820
3.10	**22.198**	**.04505**	**11.077**	**11.122**	**.99595**	**1.34631**	**1.04440**	**1.04616**	**$\bar{1}$.99824**
3.11	22.421	.04460	11.188	11.233	.99603	.35066	.04876	.05049	.99827
3.12	22.646	.04416	11.301	11.345	.99611	.35500	.05312	.05481	.99831
3.13	22.874	.04372	11.415	11.459	.99618	.35934	.05748	.05914	.99834
3.14	23.104	.04328	11.530	11.574	.99626	.36368	.06184	.06347	.99837
3.15	23.336	.04285	11.647	11.689	.99633	.36803	.06620	.06779	.99841
3.16	23.571	.04243	11.764	11.807	.99641	.37237	.07056	.07212	.99844
3.17	23.807	.04200	11.883	11.925	.99648	.37671	.07492	.07645	.99847
3.18	24.047	.04159	12.003	12.044	.99655	.38106	.07927	.08078	.99850
3.19	24.288	.04117	12.124	12.165	.99662	.38540	.08363	.08510	.99853
3.20	**24.533**	**.04076**	**12.246**	**12.287**	**.99668**	**1.38974**	**1.08799**	**1.08943**	**$\bar{1}$.99856**
3.21	24.779	.04036	12.369	12.410	.99675	.39409	.09235	.09376	.99859
3.22	25.028	.03996	12.494	12.534	.99681	.39843	.09670	.09809	.99861
3.23	25.280	.03956	12.620	12.660	.99688	.40277	.10106	.10242	.99864
3.24	25.534	.03916	12.747	12.786	.99694	.40711	.10542	.10675	.99867
3.25	25.790	.03877	12.876	12.915	.99700	.41146	.10977	.11108	.99869
3.26	26.050	.03839	13.006	13.044	.99706	.41580	.11413	.11541	.99872
3.27	26.311	.03801	13.137	13.175	.99712	.42014	.11849	.11974	.99875
3.28	26.576	.03763	13.269	13.307	.99717	.42449	.12284	.12407	.99877
3.29	26.843	.03725	13.403	13.440	.99723	.42883	.12720	.12840	.99879
3.30	**27.113**	**.03688**	**13.538**	**13.575**	**.99728**	**1.43317**	**1.13155**	**1.13273**	**$\bar{1}$.99882**
3.31	27.385	.03652	13.674	13.711	.99734	.43751	.13591	.13706	.99884
3.32	27.660	.03615	13.812	13.848	.99739	.44186	.14026	.14139	.99886
3.33	27.938	.03579	13.951	13.987	.99744	.44620	.14461	.14573	.99889
3.34	28.219	.03544	14.092	14.127	.99749	.45054	.14897	.15006	.99891
3.35	28.503	.03508	14.234	14.269	.99754	.45489	.15332	.15439	.99893
3.36	28.789	.03474	14.377	14.412	.99759	.45923	.15768	.15872	.99895
3.37	29.079	.03439	14.522	14.556	.99764	.46357	.16203	.16306	.99897
3.38	29.371	.03405	14.668	14.702	.99768	.46792	.16638	.16739	.99899
3.39	29.666	.03371	14.816	14.850	.99773	.47226	.17073	.17172	.99901
3.40	**29.964**	**.03337**	**14.965**	**14.999**	**.99777**	**1.47660**	**1.17509**	**1.17605**	**$\bar{1}$.99903**
3.41	30.265	.03304	15.116	15.149	.99782	.48094	.17944	.18039	.99905
3.42	30.569	.03271	15.268	15.301	.99786	.48529	.18379	.18472	.99907
3.43	30.877	.03239	15.422	15.455	.99790	.48963	.18814	.18906	.99909
3.44	31.187	.03206	15.577	15.610	.99795	.49397	.19250	.19339	.99911
3.45	31.500	.03175	15.734	15.766	.99799	.49832	.19685	.19772	.99912
3.46	31.817	.03143	15.893	15.924	.99803	.50266	.20120	.20206	.99914
3.47	32.137	.03112	16.053	16.084	.99807	.50700	.20555	.20639	.99916
3.48	32.460	.03081	16.215	16.245	.99810	.51134	.20990	.21073	.99918
3.49	32.786	.03050	16.378	16.408	.99814	.51569	.21425	.21506	.99919
3.50	**33.115**	**.03020**	**16.543**	**16.573**	**.99818**	**1.52003**	**1.21860**	**1.21940**	**$\bar{1}$.99921**
3.51	33.448	.02990	16.709	16.739	.99821	.52437	.22296	.22373	.99922
3.52	33.784	.02960	16.877	16.907	.99825	.52872	.22731	.22807	.99924
3.53	34.124	.02930	17.047	17.077	.99828	.53306	.23166	.23240	.99925
3.54	34.467	.02901	17.219	17.248	.99832	.53740	.23601	.23674	.99927
3.55	34.813	.02872	17.392	17.421	.99835	.54175	.24036	.24107	.99928
3.56	35.163	.02844	17.567	17.596	.99838	.54609	.24471	.24541	.99930
3.57	35.517	.02816	17.744	17.772	.99842	.55043	.24906	.24975	.99931
3.58	35.874	.02788	17.923	17.951	.99845	.55477	.25341	.25408	.99933
3.59	36.234	.02760	18.103	18.131	.99848	.55912	.25776	.25842	.99934
3.60	**36.598**	**.02732**	**18.285**	**18.313**	**.99851**	**1.56346**	**1.26211**	**1.26275**	**$\bar{1}$.99935**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
3.60	**36.598**	**.02732**	**18.285**	**18.313**	**.99851**	**1.56346**	**1 26211**	**1.26275**	**$\bar{1}$ 99935**
3 61	36.966	.02705	18 470	18.497	.99854	.56780	.26646	.26709	.99936
3.62	37.338	.02678	18 655	18.682	.99857	.57215	.27080	.27143	.99938
3.63	37 713	.02652	18 843	18 870	.99859	.57649	.27515	.27576	.99939
3.64	38 092	.02625	19 033	19.059	.99862	.58083	.27950	.28010	.99940
3 65	38 475	.02599	19 224	19 250	.99865	58517	28385	.28444	.99941
3 66	38 861	.02573	19.418	19 444	.99868	.58952	.28820	.28878	.99942
3.67	39 252	.02548	19 613	19 639	.99870	59386	.29255	.29311	.99944
3 68	39.646	.02522	19 811	19 836	.99873	59820	.29690	.29745	.99945
3.69	40 045	.02497	20.010	20 035	.99875	.60255	.30125	.30179	.99946
3.70	**40 447**	**.02472**	**20.211**	**20.236**	**.99878**	**1.60689**	**1.30559**	**1.30612**	**$\bar{1}$.99947**
3 71	40 854	.02448	20 415	20 439	.99880	.61123	.30994	.31046	.99948
3 72	41 264	.02423	20 620	20 644	.99883	.61558	.31429	.31480	.99949
3 73	41 679	.02399	20.828	20 852	.99885	61992	.31864	.31914	.99950
3.74	42 098	.02375	21.037	21 061	.99887	.62426	.32299	.32348	.99951
3.75	42 521	.02352	21.249	21 272	99889	62860	.32733	.32781	.99952
3 76	42 948	.02328	21.463	21 486	.99892	.63295	.33168	33215	.99953
3.77	43.380	.02305	21 679	21.702	.99894	.63729	.33603	.33649	.99954
3.78	43 816	.02282	21 897	21 919	.99896	.64163	.34038	.34083	.99955
3 79	44.256	.02260	22.117	22.140	.99898	.64598	.34472	.34517	.99956
3.80	**44.701**	**.02237**	**22.339**	**22.362**	**.99900**	**1.65032**	**1.34907**	**1.34951**	**$\bar{1}$.99957**
3.81	45.150	.02215	22.564	22 586	.99902	65466	.35342	.35384	.99957
3.82	45.604	.02193	22.791	22.813	.99904	.65900	.35777	.35818	.99958
3.83	46.063	.02171	23.020	23.042	.99906	.66335	.36211	.36252	.99959
3.84	46.525	.02149	23 252	23.274	.99908	66769	.36646	.36686	.99960
3.85	46.993	.02128	23.486	23.507	.99909	.67203	.37081	.37120	.99961
3.86	47.465	.02107	23.722	23.743	.99911	.67638	.37515	.37554	99961
3 87	47 942	02086	23.961	23.982	.99913	.68072	.37950	.37988	.99962
3 88	48 424	.02065	24 202	24.222	.99915	.68506	.38385	.38422	.99963
3 89	48.911	.02045	24.445	24.466	.99916	.68941	.38819	.38856	.99964
3.90	**49.402**	**.02024**	**24 691**	**24 711**	**.99918**	**1 69375**	**1 39254**	**1.39290**	**$\bar{1}$.99964**
3.91	49.899	.02004	24 939	24.960	.99920	.69809	.39689	.39724	99965
3.92	50.400	.01984	25 190	25.210	.99921	.70243	.40123	.40158	.99966
3.93	50.907	.01964	25 444	25.463	.99923	.70678	.40558	.40591	.99966
3.94	51.419	.01945	25 700	25.719	.99924	.71112	.40993	.41025	.99967
3.95	51.935	.01925	25 958	25.977	.99926	.71546	.41427	.41459	.99968
3.96	52.457	.01906	26.219	26.238	.99927	.71981	.41862	.41893	.99968
3.97	52.985	.01887	26.483	26 502	.99929	.72415	.42296	.42327	.99969
3 98	53.517	.01869	26 749	26.768	.99930	.72849	.42731	.42761	.99970
3.99	54.055	.01850	27 018	27.037	.99932	.73284	.43166	.43195	.99970
4.00	**54.598**	**.01832**	**27.290**	**27.308**	**.99933**	**1.73718**	**1.43600**	**1 43629**	**$\bar{1}$.99971**
4.01	55.147	.01813	27 564	27.583	.99934	.74152	.44035	.44063	.99971
4.02	55 701	.01795	27 842	27 860	.99936	.74586	.44469	.44497	.99972
4.03	56 261	.01777	28 122	28.139	.99937	.75021	.44904	.44931	.99973
4.04	56.826	.01760	28 404	28.422	.99938	.75455	.45339	.45365	.99973
4 05	57.397	.01742	28 690	28 707	.99939	.75889	.45773	.45799	.99974
4.06	57.974	.01725	28 979	28.996	.99941	.76324	.46208	.46233	.99974
4.07	58 557	.01708	29 270	29 287	.99942	.76758	.46642	.46668	.99975
4.08	59.145	.01691	29 564	29 581	.99943	.77192	.47077	.47102	.99975
4.09	59.740	.01674	29.862	29.878	.99944	.77626	.47511	.47536	.99976
4 10	**60.340**	**.01657**	**30.162**	**30.178**	**.99945**	**1 78061**	**1.47946**	**1.47970**	**$\bar{1}$ 99976**
4.11	60 947	.01641	30 465	30 482	.99946	.78495	.48380	.48404	.99977
4.12	61 559	.01624	30 772	30 788	.99947	.78929	.48815	.48838	99977
4.13	62 178	.01608	31.081	31.097	.99948	.79364	.49249	.49272	.99978
4.14	62.803	.01592	31.393	31.409	.99949	.79798	.49684	.49706	.99978
4.15	63.434	.01576	31.709	31.725	.99950	.80232	50118	.50140	.99978
4.16	64.072	.01561	32.028	32.044	.99951	.80667	.50553	.50574	.99979
4.17	64.715	.01545	32 350	32 365	.99952	81101	.50987	.51008	.99979
4.18	65.366	.01530	32.675	32 691	99953	.81535	.51422	.51442	.99980
4 19	66 023	.01515	33 004	33 019	.99954	.81969	.51856	.51876	.99980
4.20	**66 686**	**.01500**	**33 336**	**33 351**	**.99955**	**1.82404**	**1.52291**	**1.52310**	**$\bar{1}$.99980**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
4.20	**66.686**	**.01500**	**33.336**	**33.351**	**.99955**	**1.82404**	**1.52291**	**1.52310**	**$\bar{1}$ 99980**
4.21	67.357	.01485	33.671	33.686	.99956	.82838	.52725	.52745	.99981
4.22	68.033	.01470	34.009	34.024	.99957	.83272	.53160	.53179	.99981
4.23	68.717	.01455	34.351	34.366	.99958	.83707	.53594	.53613	.99982
4.24	69.408	.01441	34.697	34.711	.99958	.84141	.54029	.54047	.99982
4.25	70.105	.01426	35.046	35.060	.99959	.84575	.54463	.54481	.99982
4.26	70.810	.01412	35.398	35.412	.99960	.85009	.54898	.54915	.99983
4.27	71.522	.01398	35.754	35.768	.99961	.85444	.55332	.55349	.99983
4.28	72.240	.01384	36.113	36.127	.99962	.85878	.55767	.55783	.99983
4.29	72.966	.01370	36.476	36.490	.99962	.86312	.56201	.56217	.99984
4.30	**73.700**	**.01357**	**36.843**	**36.857**	**.99963**	**1.86747**	**1.56636**	**1.56652**	**$\bar{1}$ 99984**
4.31	74.440	.01343	37.214	37.227	.99964	.87181	.57070	.57086	.99984
4.32	75.189	.01330	37.588	37.601	.99965	.87615	.57505	.57520	.99985
4.33	75.944	.01317	37.966	37.979	.99965	.88050	.57939	.57954	.99985
4.34	76.708	.01304	38.347	38.360	.99966	.88484	.58373	.58388	.99985
4.35	77.478	.01291	38.733	38.746	.99967	.88918	.58808	.58822	.99986
4.36	78.257	.01278	39.122	39.135	.99967	.89352	.59242	.59256	.99986
4.37	79.044	.01265	39.515	39.528	.99968	.89787	.59677	.59691	.99986
4.38	79.838	.01253	39.913	39.925	.99969	.90221	.60111	.60125	.99986
4.39	80.640	.01240	40.314	40.326	.99969	.90655	.60546	.60559	.99987
4.40	**81.451**	**.01228**	**40.719**	**40.732**	**.99970**	**1.91090**	**1.60980**	**1.60993**	**$\bar{1}$.99987**
4.41	82.269	.01216	41.129	41.141	.99970	.91524	.61414	.61427	.99987
4.42	83.096	.01203	41.542	41.554	.99971	.91958	.61849	.61861	.99987
4.43	83.931	.01191	41.960	41.972	.99972	.92392	.62283	.62296	.99988
4.44	84.775	.01180	42.382	42.393	.99972	.92827	.62718	.62730	.99988
4.45	85.627	.01168	42.808	42.819	.99973	.93261	.63152	.63164	.99988
4.46	86.488	.01156	43.238	43.250	.99973	.93695	.63587	.63598	.99988
4.47	87.357	.01145	43.673	43.684	.99974	.94130	.64021	.64032	.99989
4.48	88.235	.01133	44.112	44.123	.99974	.94564	.64455	.64467	.99989
4.49	89.121	.01122	44.555	44.566	.99975	.94998	.64890	.64901	.99989
4.50	**90.017**	**.01111**	**45.003**	**45.014**	**.99975**	**1.95433**	**1.65324**	**1.65335**	**$\bar{1}$.99989**
4.51	90.922	.01100	45.455	45.466	.99976	.95867	.65759	.65769	.99989
4.52	91.836	.01089	45.912	45.923	.99976	.96301	.66193	.66203	.99990
4.53	92.759	.01078	46.374	46.385	.99977	.96735	.66627	.66637	.99990
4.54	93.691	.01067	46.840	46.851	.99977	.97170	.67062	.67072	.99990
4.55	94.632	.01057	47.311	47.321	.99978	.97604	.67496	.67506	.99990
4.56	95.583	.01046	47.787	47.797	.99978	.98038	.67931	.67940	.99990
4.57	96.544	.01036	48.267	48.277	.99979	.98473	.68365	.68374	.99991
4.58	97.514	.01025	48.752	48.762	.99979	.98907	.68799	.68808	.99991
4.59	98.494	.01015	49.242	49.252	.99979	.99341	.69234	.69243	.99991
4.60	**99.484**	**.01005**	**49.737**	**49.747**	**.99980**	**1.99775**	**1.69668**	**1.69677**	**$\bar{1}$.99991**
4.61	100.48	.00995	50.237	50.247	.99980	2.00210	.70102	.70111	.99991
4.62	101.49	.00985	50.742	50.752	.99981	.00644	.70537	.70545	.99992
4.63	102.51	.00975	51.252	51.262	.99981	.01078	.70971	.70979	.99992
4.64	103.54	.00966	51.767	51.777	.99981	.01513	.71406	.71414	.99992
4.65	104.58	.00956	52.288	52.297	.99982	.01947	.71840	.71848	.99992
4.66	105.64	.00947	52.813	52.823	.99982	.02381	.72274	.72282	.99992
4.67	106.70	.00937	53.344	53.354	.99982	.02816	.72709	.72716	.99992
4.68	107.77	.00928	53.880	53.890	.99983	.03250	.73143	.73151	.99993
4.69	108.85	.00919	54.422	54.431	.99983	.03684	.73577	.73585	.99993
4.70	**109.95**	**.00910**	**54.969**	**54.978**	**.99983**	**2.04118**	**1.74012**	**1.74019**	**$\bar{1}$.99993**
4.71	111.05	.00900	55.522	55.531	.99984	.04553	.74446	.74453	.99993
4.72	112.17	.00892	56.080	56.089	.99984	.04987	.74881	.74887	.99993
4.73	113.30	.00883	56.643	56.652	.99984	.05421	.75315	.75322	.99993
4.74	114.43	.00874	57.213	57.222	.99985	.05856	.75749	.75756	.99993
4.75	115.58	.00865	57.788	57.796	.99985	.06290	.76184	.76190	.99993
4.76	116.75	.00857	58.369	58.377	.99985	.06724	.76618	.76624	.99994
4.77	117.92	.00848	58.955	58.964	.99986	.07158	.77052	.77059	.99994
4.78	119.10	.00840	59.548	59.556	.99986	.07593	.77487	.77493	.99994
4.79	120.30	.00831	60.147	60.155	.99986	.08027	.77921	.77927	.99994
4.80	**121.51**	**.00823**	**60.751**	**60.759**	**.99986**	**2.08461**	**1.78355**	**1.78361**	**$\bar{1}$.99994**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
4 80	**121.51**	**.00823**	**60.751**	**60 760**	**.99986**	**2.08461**	**1.78355**	**1.78361**	**$\bar{1}$ 99994**
4 81	122 73	.00815	61.362	61 370	.99987	.08896	78790	.78796	.99994
4 82	123 97	.00807	61.979	61 987	.99987	.09330	.79224	.79230	.99994
4 83	125.21	.00799	62 601	62.609	.99987	.09764	.79658	.79664	.99994
4 84	126.47	.00791	63.231	63 239	.99987	.10199	.80093	.80098	.99995
4 85	127 74	.00783	63.866	63 874	99988	.10633	80527	.80532	99995
4 86	129.02	.00775	64.508	64.516	.99988	.11067	.80962	.80967	.99995
4.87	130.32	.00767	65 157	65 164	.99988	.11501	.81396	.81401	99995
4 88	131 63	.00760	65.812	65.819	.99988	.11936	.81830	.81835	.99995
4 89	132 95	.00752	66.473	66.481	.99989	.12370	82265	82269	.99995
4 90	**134.29**	**.00745**	**67.141**	**67 149**	**.99989**	**2.12804**	**1 82699**	**1 82704**	**$\bar{1}$ 99995**
4 91	135 64	.00737	67 816	67 823	.99989	.13239	.83133	.83138	.99995
4 92	137 00	.00730	68 498	68 505	.99989	.13673	.83568	.83572	.99995
4 93	138.38	.00723	69 186	69.193	.99990	.14107	.84002	.84006	.99995
4 94	139.77	.00715	69 882	69.889	.99990	.14541	.84436	.84441	.99996
4 95	141.17	.00708	70 584	70 591	.99990	.14976	84871	.84875	.99996
4.96	142.59	.00701	71 293	71.300	.99990	.15410	.85305	.85309	.99996
4 97	144.03	.00694	72 010	72 017	.99990	.15844	.85739	.85743	.99996
4 98	145 47	.00687	72 734	72 741	.99991	.16279	.86174	.86178	99996
4 99	146 94	.00681	73.465	73 472	.99991	.16713	.86608	.86612	.99996
5.00	**148 41**	**.00674**	**74.203**	**74 210**	**.99991**	**2.17147**	**1.87042**	**1.87046**	**$\bar{1}$ 99996**
5.01	149 90	.00667	74.949	74 956	.99991	.17582	.87477	.87480	.99996
5 02	151.41	.00660	75 702	75 710	.99991	.18016	.87911	.87915	99996
5.03	152.93	.00654	76.463	76.470	.99991	.18450	.88345	.88349	.99996
5 04	154.47	00647	77 232	77 238	.99992	.18884	.88780	.88783	.99996
5 05	156 02	.00641	78 008	78 014	.99992	.19319	.89214	.89217	.99996
5 06	157 59	.00635	78.792	78.798	.99992	.19753	89648	.89652	.99997
5 07	159 17	.00628	79 584	79 590	.99992	.20187	.90083	.90086	.99997
5 08	160 77	00622	80 384	80 390	.99992	.20622	.90517	.90520	99997
5 09	162.39	.00616	81.192	81.198	.99992	.21056	.90951	.90955	.99997
5 10	**164.02**	**.00610**	**82.008**	**82 014**	**.99993**	**2.21490**	**1.91386**	**1.91389**	**$\bar{1}$ 99997**
5.11	165 67	.00604	82 832	82 838	.99993	.21924	.91820	.91823	.99997
5.12	167.34	.00598	83.665	83.671	.99993	.22359	.92254	.92257	.99997
5.13	169.02	.00592	84.506	84 512	.99993	.22793	.92689	.92692	.99997
5 14	170 72	00586	85.355	85 361	.99993	.23227	.93123	.93126	.99997
5 15	172 43	00580	86.213	86 219	.99993	.23662	93557	.93560	.99997
5.16	174.16	.00574	87 079	87.085	.99993	.24096	.93992	.93994	.99997
5.17	175.91	.00568	87.955	87 960	.99994	.24530	.94426	.94429	.99997
5.18	177.68	.00563	88.839	88 844	.99994	.24965	.94860	.94863	.99997
5.19	179 47	.00557	89 732	89.737	.99994	.25399	.95294	.95297	.99997
5.20	**181.27**	**.00552**	**90 633**	**90.639**	**.99994**	**2.25833**	**1.95729**	**1.95731**	**$\bar{1}$.99997**
5 21	183.09	.00546	91.544	91.550	.99994	.26267	.96163	.96166	99997
5.22	184 93	.00541	92.464	92.470	.99994	.26702	.96597	.96600	.99997
5.23	186.79	.00535	93.394	93 399	.99994	.27136	.97032	.97034	.99998
5.24	188.67	.00530	94 332	94 338	.99994	.27570	.97466	.97469	.99998
5.25	190 57	.00525	95 281	95.286	.99994	.28005	.97900	.97903	.99998
5.26	192.48	.00520	96.238	96 243	.99995	.28439	98335	.98337	.99998
5 27	194 42	.00514	97 205	97.211	.99995	.28873	.98769	.98771	.99998
5 28	196 37	.00509	98.182	98 188	.99995	.29307	.99203	.99206	.99998
5.29	198.34	.00504	99.169	99.174	.99995	.29742	.99638	.99640	.99998
5.30	**200.34**	**.00499**	**100.17**	**100.17**	**.99995**	**2.30176**	**2.00072**	**2.00074**	**$\bar{1}$.99998**
5 31	202.35	.00494	101.17	101.18	.99995	.30610	.00506	.00508	.99998
5.32	204.38	.00489	102.19	102.19	99995	.31045	.00941	.00943	.99998
5.33	206.44	.00484	103.22	103.22	.99995	.31479	.01375	.01377	.99998
5.34	208 51	.00480	104.25	104.26	.99995	.31913	.01809	.01811	.99998
5.35	210 61	.00475	105.30	105 31	.99995	.32348	.02244	.02246	.99998
5.36	212.72	.00470	106.36	106 36	.99996	.32782	.02678	.02680	.99998
5 37	214.86	.00465	107.43	107.43	99996	.33216	.03112	.03114	.99998
5.38	217.02	.00461	108.51	108.51	.99996	.33650	.03547	.03548	.99998
5.39	219 20	.00456	109.60	109.60	.99996	.34085	.03981	.03983	.99998
5.40	**221.41**	**.00452**	**110.70**	**110.71**	**.99996**	**2.34519**	**2.04415**	**2.04417**	**$\bar{1}$.99998**

x	Natural Values					Common Logarithms			
	e^x	e^{-x}	Sinh x	Cosh x	Tanh x	e^x	Sinh x	Cosh x	Tanh x
5.40	**221.41**	**.00452**	**110 70**	**110.71**	**.99996**	**2.34519**	**2.04415**	**2 04417**	**$\bar{1}$.99998**
5.41	223 63	00447	111.81	111.82	.99996	.34953	.04849	.04851	.99998
5.42	225 88	.00443	112 94	112 94	.99996	.35388	.05284	.05285	.99998
5 43	228.15	.00438	114.07	114 08	.99996	.35822	.05718	.05720	.99998
5.44	230 44	00434	115.22	115 22	.99996	.36256	.06152	.06154	.99998
5.45	232.76	.00430	116 38	116 38	.99996	.36690	.06587	.06588	.99998
5.46	235.10	00425	117.55	117 55	.99996	.37125	.07021	.07023	.99998
5.47	237 46	00421	118 73	118.73	.99996	.37559	.07455	.07457	.99998
5.48	239 85	00417	119 92	119 93	.99997	.37993	.07890	.07891	.99998
5.49	242.26	00413	121.13	121.13	.99997	.38428	.08324	.08325	.99999
5 50	**244.69**	**.00409**	**122.34**	**122.35**	**99997**	**2.38862**	**2.08758**	**2.08760**	**$\bar{1}$.99999**
5 51	247.15	.00405	123.57	123.58	99997	.39296	.09193	.09194	.99999
5.52	249 64	.00401	124 82	124 82	.99997	.39731	.09627	.09628	.99999
5.53	252 14	00397	126 07	126 07	.99997	.40165	.10061	.10063	.99999
5 54	254.68	00393	127 34	127 34	.99997	.40599	.10495	.10497	.99999
5 55	257.24	.00389	128 62	128.62	.99997	.41033	.10930	.10931	.99999
5.56	259 82	00385	129 91	129.91	.99997	.41468	.11364	.11365	.99999
5.57	262 43	00381	131 22	131.22	.99997	.41902	.11798	.11800	.99999
5.58	265 07	.00377	132 53	132 54	.99997	.42336	.12233	.12234	.99999
5.59	267.74	.00374	133.87	133.87	.99997	.42771	.12667	.12668	.99999
5.60	**270.43**	**.00370**	**135.21**	**135.22**	**99997**	**2.43205**	**2.13101**	**2.13103**	**$\bar{1}$.99999**
5.61	273 14	.00366	136 57	136.57	.99997	.43639	.13536	.13537	.99999
5.62	275 89	00362	137 94	137.95	.99997	.44074	.13970	.13971	99999
5.63	278.66	.00359	139.33	139.33	.99997	.44508	.14404	.14405	.99999
5.64	281.46	.00355	140.73	140 73	.99997	.44942	.14839	.14840	.99999
5.65	284.29	.00352	142 14	142 15	.99998	.45376	.15273	.15274	.99999
5.66	287.15	00348	143.57	143.58	.99998	.45811	.15707	.15708	.99999
5.67	290 03	.00345	145.02	145.02	.99998	.46245	16141	.16142	.99999
5.68	292 95	.00341	146.47	146 48	.99998	.46679	.16576	.16577	99999
5.69	295.89	.00338	147.95	147.95	.99998	.47114	.17010	.17011	.99999
5.70	**298 87**	**.00335**	**149.43**	**149.44**	**.99998**	**2.47548**	**2.17444**	**2.17445**	**$\bar{1}$.99999**
5 71	301.87	.00331	150 93	150.94	.99998	.47982	.17879	.17880	.99999
5.72	304.90	.00328	152 45	152.45	99998	.48416	.18313	.18314	.99999
5.73	307.97	.00325	153 98	153 99	.99998	.48851	.18747	.18748	.99999
5.74	311.06	.00321	155.53	155 53	.99998	.49285	.19182	.19182	.99999
5.75	314.19	.00318	157 09	157.10	.99998	.49719	.19616	.19617	.99999
5.76	317.35	.00315	158.67	158.68	.99998	.50154	.20050	.20051	.99999
5.77	320.54	.00312	160.27	160.27	.99998	.50588	.20484	.20485	.99999
5.78	323 76	.00309	161.88	161.88	.99998	.51022	.20919	.20920	.99999
5.79	327.01	.00306	163.51	163.51	.99998	.51457	.21353	.21354	.99999
5.80	**330.30**	**.00303**	**165.15**	**165.15**	**.99998**	**2.51891**	**2.21787**	**2.21788**	**$\bar{1}$.99999**
5.81	333 62	.00300	166.81	166 81	.99998	.52325	.22222	.22222	.99999
5.82	336.97	.00297	168.48	168.49	99998	.52759	.22656	22657	.99999
5 83	340 36	.00294	170.18	170.18	.99998	.53194	.23090	.23091	.99999
5.84	343.78	.00291	171.89	171.89	.99998	.53628	.23525	.23525	99999
5.85	347.23	.00288	173 62	173 62	.99998	.54062	23959	.23960	.99999
5.86	350.72	.00285	175.36	175 36	.99998	.54497	.24393	.24394	.99999
5.87	354.25	.00282	177.12	177.13	.99998	.54931	.24828	.24828	.99999
5.88	357.81	.00279	178 90	178.91	.99998	.55365	.25262	.25262	99999
5.89	361.41	.00277	180.70	180.70	.99998	.55799	.25696	.25697	.99999
5.90	**365.04**	**.00274**	**182.52**	**182.52**	**.99998**	**2 56234**	**2 26130**	**2.26131**	**$\bar{1}$.99999**
5.91	368.71	.00271	184.35	184.35	.99999	.56668	.26565	.26565	.99999
5.92	372.41	.00269	186.20	186 21	.99999	.57102	.26999	.27000	.99999
5.93	376.15	.00266	188.08	188.08	.99999	.57537	.27433	.27434	.99999
5.94	379.93	.00263	189.97	189.97	.99999	.57971	.27868	.27868	.99999
5.95	383.75	.00261	191.88	191 88	.99999	.58405	.28302	.28303	.99999
5.96	387.61	.00258	193.80	193.81	.99999	.58840	.28736	.28737	.99999
5.97	391.51	.00255	195 75	195 75	.99999	59274	.29171	.29171	.99999
5.98	395.44	.00253	197 72	197.72	.99999	.59708	.29605	.29605	.99999
5.99	399.41	.00250	199.71	199.71	.99999	.60142	.30039	.30040	.99999
6.00	**403.43**	**.00248**	**201.71**	**201 72**	**.99999**	**2 60577**	**2 30473**	**2.30474**	**$\bar{1}$ 99999**

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